

MODULES ATTACHED TO EXTENSION BUNDLES

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ABSTRACT. In this article we study modules over wild canonical algebras which correspond to extension bundles [9] over weighted projective lines. We prove that all modules attached to extension bundles can be established by matrices with coefficients related to the relations of the considered algebra.

Moreover, we expand the concept of extension bundles over weighted projective lines with three weights to general weight type and establish similar results in this situation. Finally, we present a method to compute matrices for all modules attached to extension bundles using cokernels of maps between direct sums of line bundles.

1. INTRODUCTION

One of the problems of representation theory of finite-dimensional algebras is the classification of indecomposable modules over a given algebra. Depending on the complexity of this issue we distinguish algebras of finite, tame and wild representation type. In the case of wild algebras the structure of the module category is rich enough that it is impossible to describe all indecomposable modules, however in this situation sometimes it is possible to describe subclasses of indecomposable modules.

In this paper we study an important class of modules, namely the so called extension modules for wild canonical algebras. Canonical algebras were introduced by C. M. Ringel in 1984 [14], for a definition we refer to Section 2.

In the case of domestic canonical algebras D. Kussin and the second author [11] described matrices for all indecomposable modules provided the characteristic of the field is different from 2. In the case of characteristic 2 matrices for the indecomposables were given in [8].

In the situation of tubular canonical algebras in [13], extending methods of [15], it was shown that the exceptional modules can be exhibited by matrices having as coefficients only 0, 1 and -1 in the cases $(3, 3, 3)$, $(2, 4, 4)$, $(2, 3, 6)$ and 0, 1, -1 , λ , $\lambda - 1$ in the case $(2, 2, 2, 2)$, where λ is the parameter appearing in the relations for the considered algebra. Based on this result in [1] an algorithm and a computer program were developed to determine a description of all exceptional modules over tubular canonical algebras. Further in [3] and [2] the problem of homogeneous modules over tubular canonical modules was studied, in particular explicit matrices for modules of integral slope were given.

For canonical algebras of wild type it was proved by the authors in [7] that "almost all" exceptional modules can be described by matrices having coefficients $\lambda_i - \lambda_j$, where the λ_i are the parameters of the canonical algebra.

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In [5] W. Geigle and H. Lenzing investigated weighted projective lines to give a geometric approach to canonical algebras. More precisely, they showed that the category of coherent sheaves $\text{coh}(\mathbb{X})$ over a weighted projective line \mathbb{X} admits a tilting bundle T which induces an equivalence of the bounded derived categories $D^b(\text{coh}(\mathbb{X})) \cong D^b(\text{mod}(\Lambda))$, where $\Lambda = \text{End}_\Lambda(T)$ is a canonical algebra.

In 2013 D. Kussin, H. Lenzing and second author in [9] introduced the concept of extension bundles over weighted projective lines with three weights, which is important in the study of nilpotent operators with invariant subspaces (see also [10]). It was proved there, in particular, that each indecomposable vector bundle of rank two is exceptional and appears as the middle term of an exact sequence, where the other terms are line bundles with good homological properties, see Section 3.

The aim of this article is to study modules attached to such extension bundles in the case of canonical algebras of wild type. Those modules are called *extension modules*. The paper contains the following results.

1. We prove that all extension modules over a wild canonical algebra Λ with three arms, can be described by matrices with entries 0, 1 and -1 . This is an improvement for those modules of the result from [7]. We will use the fact that the category of vector bundles $\text{vect}(\mathbb{X})$ over a weighted projective line \mathbb{X} is a Frobenius category with the line bundles as the indecomposable projective-injective objects. The main tool in the proof is the fact that a vector bundle associated to a module has a line bundle, associated to a module, as a direct summand of its projective cover.
2. We extend the concept of extension bundles from [9] to the case of an arbitrary number of weights. If this number is greater than 3, then not every indecomposable vector bundle of rank two is exceptional. We present a useful characterization of exceptional modules of rank two as extension bundles with data (L, \vec{x}) , where L is a line bundle and \vec{x} is an element of the grading group of a specific form. We also establish the projective covers and the injective hulls of those bundles.
3. We show that all extension modules for a wild canonical algebra with an arbitrary number of arms can be established by matrices with coefficients 0, λ_i , $-\lambda_i$ where the λ_i are the parameters of the canonical algebra.
4. We compute matrix representation for each extension module over a canonical algebra of arbitrary type. Since the method using Schofield induction for exceptional modules (see [15], [6]) is not constructive we can not proceed as in the case for tubular canonical algebras [13]. Therefore here we present another idea. We show that each extension module appears as a cokernel of a map between direct sums of line bundles and we describe a method to calculate matrices for these cokernels.

2. NOTATIONS AND BASIC CONCEPTS

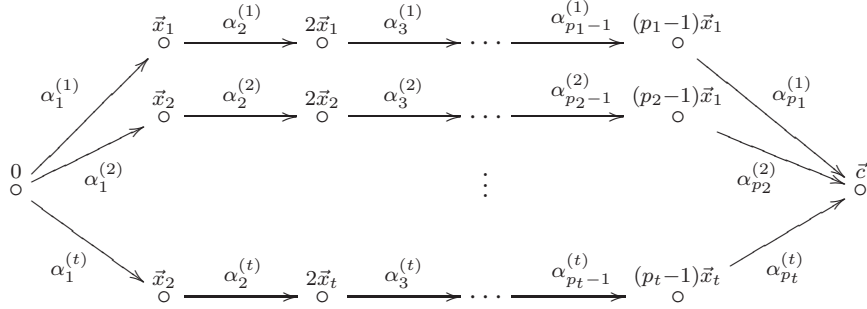
Let k be an algebraically closed field. We recall the concept of a weighted projective line in the sense of W. Geigle and H. Lenzing [5]. Let $\mathbb{L} = \mathbb{L}(\underline{p})$ be the rank one abelian group with generators $\vec{x}_1, \dots, \vec{x}_t$ and relations $p_1\vec{x}_1 = \dots = p_t\vec{x}_t := \vec{c}$, where the p_i are integers greater than or equal to 2. These numbers are called *weights*. The element \vec{c} is called the *canonical element*. Recall that \mathbb{L} is an ordered group with $\mathbb{L}_+ = \sum_{i=1}^t \mathbb{N}\vec{x}_i$ as its set of non-negative elements. Moreover, each element \vec{y} of \mathbb{L} can be written in *normal form* $\vec{y} = a\vec{c} + \sum_{i=1}^t a_i\vec{x}_i$ with $a \in \mathbb{Z}$ and $0 \leq a_i < p_i$. The polynomial algebra $k[x_1, \dots, x_t]$ is \mathbb{L} -graded, where the degree of

x_i is \vec{x}_i . Since the polynomials $f_i = x_i^{p_i} - x_1^{p_1} - \lambda_i x_2^{p_2}$ for $i = 3, \dots, t$ are homogeneous, the quotient algebra $S = k[x_1, \dots, x_t] / \langle f_i \mid i = 3, \dots, t \rangle$ is also \mathbb{L} -graded. Here the λ_i are pairwise distinct non-zero elements of k , they are called the *parameters*. A *weighted projective line* \mathbb{X} is the projective spectrum of the \mathbb{L} -graded algebra S . Therefore \mathbb{X} depends on a weight sequence $\underline{p} = (p_1, \dots, p_t)$ and a sequence of parameters $\underline{\lambda} = (\lambda_3, \dots, \lambda_t)$. We can assume that $\lambda_3 = 1$. The category of coherent sheaves over \mathbb{X} will be denoted by $\text{coh}(\mathbb{X})$. Each indecomposable sheaf in $\text{coh}(\mathbb{X})$ is a locally free sheaf, called a *vector bundle*, or a *sheaf of finite length*. Denote by $\text{vect}(\mathbb{X})$ (resp. $\text{coh}_0(\mathbb{X})$) the subcategory of $\text{coh}(\mathbb{X})$ consisting of all vector bundles (resp. finite length sheaves) on \mathbb{X} .

The category $\text{coh}(\mathbb{X})$ is a Hom-finite, abelian k -category. Moreover, it is hereditary that is $\text{Ext}_{\mathbb{X}}^i(-, -) = 0$ for $i \geq 2$ and has Serre duality in the form $\text{Ext}_{\mathbb{X}}^1(F, G) \cong D\text{Hom}_{\mathbb{X}}(G, \tau_{\mathbb{X}}F)$, where the Auslander-Reiten translation $\tau_{\mathbb{X}}$ is given by the shift $F \mapsto F(\vec{\omega})$, where $\vec{\omega} := (t-2)\vec{c} - \sum_{i=1}^t \vec{x}_i$ denotes the *dualizing element*. It is well known that each line bundle has the form $\mathcal{O}(\vec{x})$ where \mathcal{O} is the structure sheaf of \mathbb{X} and where $\vec{x} \in \mathbb{L}$. Furthermore we have isomorphisms $\text{Hom}_{\mathbb{X}}(\mathcal{O}(\vec{x}), \mathcal{O}(\vec{y})) \cong S_{\vec{y}-\vec{x}}$, where $S_{\vec{z}}$ denotes the grading component of S associated to $\vec{z} \in \mathbb{L}$.

One of the main results proved in [5] is the fact that in $\text{coh}(\mathbb{X})$ there is a tilting object, which is a direct sum of line bundles $T = \bigoplus_{0 \leq \vec{x} \leq \vec{c}} \mathcal{O}(\vec{x})$, such that the right derived functor of the functor $\text{Hom}_{\mathbb{X}}(T, -)$ induces an equivalence of bounded derived category $\mathcal{D}^b(\text{coh}(\mathbb{X})) \xrightarrow{\cong} \mathcal{D}^b(\text{mod}(\Lambda))$, where $\Lambda = \text{End}_{\mathbb{X}}(T)$ is a canonical algebra, called the *canonical algebra* associated to the weighted projective line \mathbb{X} .

Originally, canonical algebras Λ were introduced by C. M. Ringel [14] as path algebras of quivers Q :



with *canonical relations*

$$\alpha_{p_i}^{(i)} \dots \alpha_2^{(i)} \alpha_1^{(i)} = \alpha_{p_1}^{(1)} \dots \alpha_2^{(1)} \alpha_1^{(1)} + \lambda_i \alpha_{p_2}^{(2)} \dots \alpha_2^{(2)} \alpha_1^{(2)} \quad \text{for } i = 3, 4, \dots, t,$$

where the λ_i are parameters from $\underline{\lambda}$ and p_i are weights from \underline{p} as before. We call t the number of arms of Λ . Concerning the complexity of the module category over Λ there are three types of canonical algebras, domestic, tubular and wild ones. Recall that Λ is of domestic (respectively tubular, wild) type if the Euler characteristic $\chi_{\Lambda} = (2-t) + \sum_{i=1}^t 1/p_i$ is positive (respectively zero, negative).

Denote by Q_0 the set of vertices and by Q_1 the set of arrows of the quiver Q . Then each finitely generated right module over Λ is given by finite dimensional vector spaces M_i for each vertex i of Q_0 and by linear maps $M_{\alpha} : M_j \rightarrow M_i$ for each arrows $\alpha : i \rightarrow j$ of Q_1 such that the canonical relations are satisfied. We will usually identify linear maps with matrices. The category of finite generated right modules we denote by $\text{mod}(\Lambda)$.

For coherent sheaves there are well known invariants the *rank*, the *degree* and the *determinant*, which are given by linear forms on the Grothendieck group $\mathrm{rk}, \mathrm{deg} : K_0(\mathbb{X}) \rightarrow \mathbb{Z}$ and $\mathrm{det} : K_0(\mathbb{X}) \rightarrow \mathbb{L}(\underline{p})$. Since $K_0(\mathbb{X}) \simeq K_0(\Lambda)$ we have also the concept of the rank, the degree and the determinants for Λ -modules. In particular the rank of a Λ -module is defined by the formula $\mathrm{rk}M := \dim_k M_0 - \dim_k M_{\vec{c}}$. We denote by $\mathrm{mod}_+(\Lambda)$ (respectively $\mathrm{mod}_-(\Lambda)$, $\mathrm{mod}_0(\Lambda)$) the full subcategory consisting of all Λ -modules, whose indecomposable summands in the decomposition into a direct sum have positive (respectively negative or zero) rank. Further, by $\mathrm{coh}_+(\mathbb{X})$ (resp. $\mathrm{coh}_-(\mathbb{X})$) we denote the full subcategory of all vector bundles X on \mathbb{X} , such that the functor $\mathrm{Ext}_{\mathbb{X}}^1(T, X) = 0$ (resp. $\mathrm{Hom}_{\mathbb{X}}(T, X) = 0$). Under the equivalence $\mathcal{D}^b(\mathrm{coh}(\mathbb{X})) \xrightarrow{\cong} \mathcal{D}^b(\mathrm{mod}(\Lambda))$

- $\mathrm{coh}_+(\mathbb{X})$ corresponds to $\mathrm{mod}_+(\Lambda)$ by means of $E \mapsto \mathrm{Hom}_{\mathbb{X}}(T, E)$,
- $\mathrm{coh}_0(\mathbb{X})$ corresponds to $\mathrm{mod}_0(\Lambda)$ by means of $E \mapsto \mathrm{Hom}_{\mathbb{X}}(T, E)$,
- $\mathrm{coh}_-(\mathbb{X})[1]$ corresponds to $\mathrm{mod}_-(\Lambda)$ by means of $E[1] \mapsto \mathrm{Ext}_{\mathbb{X}}^1(T, E)$, where $[1]$ denotes the suspension functor of the triangulated category $\mathcal{D}^b(\mathrm{coh}(\mathbb{X}))$.

We say in these cases that the module $\mathrm{Hom}_{\mathbb{X}}(T, E)$ (respectively $\mathrm{Ext}_{\mathbb{X}}^1(T, E)$) is *attached* to E . For simplicity we will often identify a sheaf E in $\mathrm{coh}_+(\mathbb{X})$ or $\mathrm{coh}_0(\mathbb{X})$ with the corresponded Λ -module $\mathrm{Hom}_{\mathbb{X}}(T, E)$.

Remark 2.1. Recall from [4, Theorem 9.1.1] that the standard duality $\mathrm{Hom}_{\Lambda}(-, k)$ defines an equivalence of the categories $\mathrm{mod}(\Lambda)$ and $\mathrm{mod}(\Lambda^{op})$. Under this equivalence $\mathrm{mod}_-(\Lambda)$ corresponds to $\mathrm{mod}_+(\Lambda^{op})$. Since $\Lambda \simeq \Lambda^{op}$ in many considerations it is sufficient to consider Λ -modules of positive rank and of rank zero. In particular, modules of negative rank can be obtained from those of positive rank by reversing the arrows and transposing the matrices.

Recall that a coherent sheaf E over \mathbb{X} is called *exceptional* if $\mathrm{End}_{\mathbb{X}}(E) = k$ and $\mathrm{Ext}_{\mathbb{X}}^1(E, E) = 0$. A pair (X, Y) in $\mathrm{coh}(\mathbb{X})$ is called an *exceptional pair* if X, Y are exceptional and $\mathrm{Hom}_{\mathbb{X}}(Y, X) = 0 = \mathrm{Ext}_{\mathbb{X}}^1(Y, X)$. Furthermore, an exceptional pair is *orthogonal* if addition $\mathrm{Hom}_{\mathbb{X}}(X, Y) = 0$. Finally, a Λ -module M is called *exceptional* if $\mathrm{End}_{\Lambda}(M) = k$ and $\mathrm{Ext}_{\Lambda}^i(M, M) = 0$ for $i \geq 1$.

3. EXTENSION BUNDLES FOR WEIGHTED PROJECTIVE LINES WITH THREE WEIGHTS

Let \mathbb{X} be a weighted projective line of a triple type (p_1, p_2, p_3) . The concept of extension bundles was introduced in [9] in the study of stable vector bundle categories. In particular it was shown that stable vector bundle categories of weighted projective lines of triple type admit tilting objects, being direct sums of extension bundles, such that their endomorphism algebras form cuboids.

From [9, Theorem 4.2.] each indecomposable vector bundle E can be obtained as the middle term of a non-split exact sequence

$$\eta_{L, \vec{x}} : 0 \rightarrow L(\vec{\omega}) \rightarrow E \rightarrow L(\vec{x}) \rightarrow 0$$

for some line bundle L and some element \vec{x} of \mathbb{L} such that $0 \leq \vec{x} \leq \vec{\delta}$, where $\vec{\delta} := \vec{c} + 2\vec{\omega}$ is the *dominant element*. Because in this case the vector space $\mathrm{Ext}_{\mathbb{X}}^1(L(\vec{x}), L(\vec{\omega}))$ is one-dimensional the bundle E is uniquely determined up to isomorphism. It is called the *extension bundle* given by the pair (L, \vec{x}) . It is easy to check, that the pair $(L(\vec{x}), L(\vec{\omega}))$ is exceptional and orthogonal. Therefore, if $L(\vec{x})$ and $L(\vec{\omega})$ are Λ -modules, both in $\mathrm{mod}_+(\Lambda)$ or in $\mathrm{mod}_-(\Lambda)$, then they can

be described by $0, \pm\lambda_i$ matrices, as rank one modules (see [13]) and it follows that E also can be described by matrices with the same coefficients (see Proposition 7.1 and the remark after its proof in [7]).

We recall that the category $\text{vect}(\mathbb{X})$ of vector bundles over \mathbb{X} is a Frobenius category such that the indecomposable projective-injective objects are exactly the line bundles [9, Def. 3.1]. Moreover each vector bundle has a projective cover and an injective hull.

Lemma 3.1. *Let E be a non-zero vector bundle on a weighted projective line \mathbb{X} of type (p_1, p_2, p_3) with projective cover $\mathfrak{P}(E)$. If $\text{Ext}_{\mathbb{X}}^1(T, E) = 0$ for the canonical bundle T , then there is a line bundle L in the decomposition $\mathfrak{P}(E)$ into a direct sum of line bundles, such that $\text{Ext}_{\mathbb{X}}^1(T, L) = 0$.*

Proof. Assume that $\text{Ext}_{\mathbb{X}}^1(T, E) = 0$. Then $\text{Hom}_{\mathbb{X}}(T, E) \neq 0$ because T is a tilting bundle and E is non-zero. Therefore there is an element \vec{x} , such that $0 \leq \vec{x} \leq \vec{c}$ and $\text{Hom}_{\mathbb{X}}(\mathcal{O}(\vec{x}), E) \neq 0$. Each non-zero morphism $f : \mathcal{O}(\vec{x}) \rightarrow E$ factors through $\pi_E : \mathfrak{P}(E) \rightarrow E$, so there is morphism $0 \neq \alpha : \mathcal{O}(\vec{x}) \rightarrow \mathfrak{P}(E)$ such that $\pi_E \circ \alpha = f$. Hence there is a direct summand $L = \mathcal{O}(\vec{y})$ of $\mathfrak{P}(E)$ such that $\text{Hom}_{\mathbb{X}}(\mathcal{O}(\vec{x}), \mathcal{O}(\vec{y})) \neq 0$.

We will show that L has the desired property. Writing \vec{y} in normal form $\vec{y} = \alpha\vec{c} + \alpha_1\vec{x}_1 + \alpha_2\vec{x}_2 + \alpha_3\vec{x}_3$, with $0 \leq \alpha_i \leq p_i - 1$, we obtain $\alpha \geq 0$ and

$$(\Delta) \quad \vec{c} + \vec{\omega} - \vec{y} = (2 - \alpha)\vec{c} - \sum_{i=1}^3 (\alpha_i + 1)\vec{x}_i < 0.$$

Assume that $\text{Ext}_{\mathbb{X}}^1(T, \mathcal{O}(\vec{y})) \neq 0$ and let \vec{z} satisfy $\text{Ext}_{\mathbb{X}}^1(\mathcal{O}(\vec{z}), \mathcal{O}(\vec{y})) \neq 0$ and $0 \leq \vec{z} \leq \vec{c}$. Using Serre duality we obtain $\vec{z} + \vec{\omega} - \vec{y} \geq 0$. Therefore

$$\vec{c} + \vec{\omega} - \vec{y} = \underbrace{(\vec{c} - \vec{z})}_{\geq 0} + \underbrace{(\vec{z} + \vec{\omega} - \vec{y})}_{\geq 0} \geq 0,$$

a contradiction with (Δ) . Thus $\text{Ext}_{\mathbb{X}}^1(T, \mathcal{O}(\vec{y})) = 0$ \square

In the following lemma we prove that each extension bundle defined by a short exact sequence $\eta_{L, \vec{x}}$ appears in addition as an extension bundle for three different pairs (L, \vec{x}) .

Lemma 3.2. *Let \mathbb{X} be a weighted projective line of type (p_1, p_2, p_3) and let E be an extension bundle given by a pair (L, \vec{x}) , where $\vec{x} = l_1\vec{x}_1 + l_2\vec{x}_2 + l_3\vec{x}_3$. Then E is also an extension bundle determined by the following pairs:*

$$\left(L(\vec{x} - (1 + l_i)\vec{x}_i)(-\vec{\omega}), \quad l_i\vec{x}_i + \sum_{j \neq i} (p_j - l_j - 2)\vec{x}_j \right) \quad \text{for } i = 1, 2, 3,$$

where $L(\vec{x} - (1 + l_i)\vec{x}_i)$ are direct summands of $\mathfrak{P}(E)$. In particular, for each $i \in \{1, 2, 3\}$, there is an exact short sequence

$$0 \longrightarrow L(\vec{x} - (1 + l_i)\vec{x}_i) \longrightarrow E \longrightarrow L((1 + l_i)\vec{x}_i + \vec{\omega}) \longrightarrow 0.$$

Proof. The projective cover of E has the form $\mathfrak{P}(E) = L(\vec{\omega}) \oplus \bigoplus_{i=1}^3 L(\vec{x} - (1 + l_i)\vec{x}_i)$, [9, Theorem 4.6.]. Then there are exact sequences

$$\eta_i : \quad 0 \longrightarrow L(\vec{x} - (1 + l_i)\vec{x}_i) \longrightarrow E \longrightarrow \widehat{L}_i \longrightarrow 0, \quad \text{for } i = 1, 2, 3,$$

where from [9, Proposition 3.8] the sheaf \widehat{L}_i is a line bundle, for $i = 1, 2, 3$. By comparison of the determinants we obtain that $\widehat{L}_i = L((1 + l_i)\vec{x}_i + \vec{\omega})$ is a direct

summand of the injective hull of E . Therefore the sequence η_i can be presented as follows

$$0 \longrightarrow L(\vec{x} - (1 + l_i)\vec{x}_i - \vec{\omega})(\vec{\omega}) \longrightarrow E \longrightarrow L(\vec{x} - (1 + l_i)\vec{x}_i - \vec{\omega})(\vec{y}_i) \longrightarrow 0,$$

where $\vec{y}_i = (1 + l_i)\vec{x}_i + 2\vec{\omega} - \vec{x} = l_i\vec{x}_i + \sum_{j \neq i} (p_j - l_j - 2)\vec{x}_j$. Since $0 \leq l_j \leq p_j - 2$ we have $0 \leq p_j - l_j - 2 \leq p_j - 2$, and consequently $0 \leq \vec{y}_i \leq \vec{\delta}$. \square

Theorem 1. *Let Λ be a canonical algebra with three arms. Then each indecomposable Λ -module of rank two can be described by matrices having coefficients 0, 1, -1 .*

Proof. Let M be a Λ -module of rank 2, attached to an indecomposable vector bundle E over the weighted projective line \mathbb{X} associated to Λ . Then M is in $\text{mod}_+(\Lambda)$. We will show that there is an exact sequence

$$\eta: \quad 0 \longrightarrow A \longrightarrow E \longrightarrow B \longrightarrow 0,$$

where $A, B \in \text{mod}_+(\Lambda)$ and (B, A) is an orthogonal exceptional pair in $\text{coh}(\mathbb{X})$. Then the result follows from [7, Proposition 7.1].

From [9, Theorem 4.2] the vector bundle E appears as an extension bundle given by a pair (L, \vec{x}) , this means that there is a short exact sequence

$$\eta_{L, \vec{x}}: \quad 0 \longrightarrow L(\vec{\omega}) \longrightarrow E \longrightarrow L(\vec{x}) \longrightarrow 0,$$

where $(L(\vec{x}), L(\vec{\omega}))$ is an orthogonal exceptional pair. Since the vector bundle E is attached to the Λ -module M we have $\text{Ext}_{\mathbb{X}}^1(T, E) = 0$. Applying the functor $\text{Hom}_{\mathbb{X}}(T, -)$ to the sequence $\eta_{L, \vec{x}}$ we obtain that $\text{Ext}_{\mathbb{X}}^1(T, L(\vec{x})) = 0$, thus $L(\vec{x})$ is in $\text{mod}_+(\Lambda)$. If in addition $\text{Ext}_{\mathbb{X}}^1(T, L(\vec{\omega})) = 0$ we are done. Otherwise we will replace the exact sequence $\eta_{L, \vec{x}}$ by another one.

To do so we recall that $L(\vec{\omega})$ is a direct summand of the projective cover $\mathfrak{P}(E)$ and from Lemma 3.1 there is a line bundle \widehat{L} , which is a direct summand of $\mathfrak{P}(E)$ and satisfies $\text{Ext}_{\mathbb{X}}^1(T, \widehat{L}) = 0$ thus \widehat{L} is in $\text{coh}_+(\mathbb{X}) = \text{mod}_+(\Lambda)$. From [9, Theorem 4.6] the line bundle \widehat{L} has the form $L(\vec{x} - (1 + l_i)\vec{x}_i)$ for some $i \in \{1, 2, 3\}$. Thus using Lemma 3.2 we get an exact sequence

$$0 \longrightarrow L(\vec{x} - (1 + l_i)\vec{x}_i) \longrightarrow E \longrightarrow L((1 + l_i)\vec{x}_i + \vec{\omega}) \longrightarrow 0.$$

Applying the functor $\text{Hom}_{\mathbb{X}}(T, -)$ to the sequence above we conclude that $\text{Ext}_{\mathbb{X}}^1(E, L((1 + l_i)\vec{x}_i + \vec{\omega})) = 0$ and therefore $L((1 + l_i)\vec{x}_i + \vec{\omega})$ is in $\text{mod}_+(\Lambda)$. Moreover it is easily checked that $(L(\vec{x} - (1 + l_i)\vec{x}_i), L((1 + l_i)\vec{x}_i + \vec{\omega}))$ form an orthogonal exceptional pair in $\text{coh}(\mathbb{X})$. Thus we get an exact sequence η of the desired form and the theorem is proved. \square

Remark 3.3. Using Remark 2.1 we get the same result for exceptional modules of rank -2 from $\text{mod}_-(\Lambda)$.

4. EXTENSION BUNDLES IN THE CASE OF t NUMBERS OF WEIGHTS

In this section we will deal with a weighted projective line \mathbb{X} of the type (p_1, \dots, p_t) where t is greater than or equal to 3.

Theorem 2. *Let \mathbb{X} be a weighted projective line of a type (p_1, \dots, p_t) . Then each indecomposable vector bundle of rank two occurs as the middle term of a non-split exact sequence*

$$\eta: \quad 0 \longrightarrow L(\vec{\omega}) \xrightarrow{i} E \xrightarrow{\pi} L(\vec{x}) \longrightarrow 0,$$

where $0 \leq \vec{x} \leq \vec{\delta} := 2\vec{\omega} + \vec{c} = (t-3)\vec{c} + \sum_{i=1}^t (p_i - 2)\vec{x}_i$. Moreover the following conditions are equivalent:

- (i) The vector bundle E is exceptional.
- (ii) The pair $(L(\vec{x}), L(\vec{\omega}))$ is an orthogonal exceptional pair with $\text{Ext}_{\mathbb{X}}^1(L(\vec{x}), L(\vec{\omega})) = k$.
- (iii) $\vec{x} = \sum_{i=1}^t l_i \vec{x}_i$ with $0 \leq l_i \leq p_i - 1$ and there are exactly $t-3$ numbers l_i equal to $p_i - 1$.
- (iv) $\text{Ext}_{\mathbb{X}}^1(L(\vec{\omega}), L(\vec{x})) = 0$, $\text{Hom}_{\mathbb{X}}(E, L(\vec{\omega})) = 0 = \text{Ext}_{\mathbb{X}}^1(E, L(\vec{\omega}))$ and $\text{Hom}_{\mathbb{X}}(L(\vec{x}), E) = 0 = \text{Ext}_{\mathbb{X}}^1(L(\vec{x}), E)$.

Proof. The proof of the existence of the sequence η is almost the same as in the case of a triple weight type, we refer the reader to [9, Theorem 4.2].

(i) \Rightarrow (ii). Assume that E is exceptional and $0 \leq \vec{x} \leq \vec{\delta}$. Then \vec{x} can be written in normal form $\vec{x} = l\vec{c} + \sum_{i=1}^t l_i \vec{x}_i$, with $l \geq 0$ and $0 \leq l_i \leq p_i - 1$ and $\vec{x} \leq \vec{\delta}$. Because $\vec{x} - \vec{\omega} \leq \vec{\delta} - \vec{\omega} = \vec{\omega} - \vec{c} < 0$ we have $\text{Hom}_{\mathbb{X}}(L(\vec{\omega}), L(\vec{x})) \cong S_{\vec{x}-\vec{\omega}} = 0$. Similarly, the vector space $\text{Hom}_{\mathbb{X}}(L(\vec{x}), L(\vec{\omega}))$ also vanishes. From Serre duality we get

$$\text{Ext}_{\mathbb{X}}^1(L(\vec{x}), L(\vec{\omega})) \cong \text{Hom}_{\mathbb{X}}(L(\vec{\omega}), L(\vec{x} + \vec{\omega})) \cong S_{\vec{x}} \cong k^{l+1}.$$

Now, we have $[E] = [L(\vec{\omega})] + [L(\vec{x})]$ in the Grothendieck group $K_0(\mathbb{X})$ and applying the Euler form $\langle -, - \rangle_{\mathbb{X}} : K_0(\mathbb{X}) \times K_0(\mathbb{X}) \rightarrow \mathbb{Z}$ we obtain

$$\begin{aligned} 1 &= \langle [E], [E] \rangle_{\mathbb{X}} = \langle [L(\vec{\omega})] + [L(\vec{x})], [L(\vec{\omega})] + [L(\vec{x})] \rangle_{\mathbb{X}} \\ &= 2 + \dim_k \text{Hom}_{\mathbb{X}}(L(\vec{\omega}), L(\vec{x})) - \dim_k \text{Ext}_{\mathbb{X}}^1(L(\vec{\omega}), L(\vec{x})) \\ &\quad + \dim_k \text{Hom}_{\mathbb{X}}(L(\vec{x}), L(\vec{\omega})) - \dim_k \text{Ext}_{\mathbb{X}}^1(L(\vec{x}), L(\vec{\omega})) \\ &= 2 - (l+1) - \dim_k \text{Ext}_{\mathbb{X}}^1(L(\vec{\omega}), L(\vec{x})). \end{aligned}$$

Therefore $-l = \dim_k \text{Ext}_{\mathbb{X}}^1(L(\vec{\omega}), L(\vec{x}))$ and it follows that $l \geq 0$. Consequently $\dim_k \text{Ext}_{\mathbb{X}}^1(L(\vec{\omega}), L(\vec{x})) = 0$ and $\text{Ext}_{\mathbb{X}}^1(L(\vec{x}), L(\vec{\omega})) = k$.

(ii) \Rightarrow (iii). Assume that $(L(\vec{x}), L(\vec{\omega}))$ is an orthogonal exceptional pair, such that $\text{Ext}_{\mathbb{X}}^1(L(\vec{x}), L(\vec{\omega})) \cong k$. The element \vec{x} can be written in normal form $\vec{x} = l\vec{c} + \sum_{i=1}^t l_i \vec{x}_i$, with $l \geq 0$ and $0 \leq l_i \leq p_i - 1$. From Serre duality we obtain that $\text{Ext}_{\mathbb{X}}^1(L(\vec{x}), L(\vec{\omega})) \cong k^{l+1}$. Hence $l = 0$. Moreover

$$0 = \dim_k \text{Ext}_{\mathbb{X}}^1(L(\vec{\omega}), L(\vec{x})) \cong \text{DHom}_{\mathbb{X}}(L(\vec{x}), L(2\vec{\omega})) \cong S_{2\vec{\omega}-\vec{x}},$$

where $2\vec{\omega} - \vec{x} = (t-4)\vec{c} + \sum_{i=1}^t (p_i - 2 - l_i)\vec{x}_i < 0$. Therefore at least $t-3$ numbers l_i have to be equal to $p_i - 1$. Furthermore, because $\vec{x} \leq \vec{\delta}$ at most $t-3$ numbers l_i can be equal to $p_i - 1$. This implies that exactly $t-3$ numbers l_i are equal to $p_i - 1$.

(iii) \Rightarrow (iv). Assume that the element \vec{x} has normal form $\sum_{i=1}^t l_i \vec{x}_i$ with $0 \leq l_i \leq p_i - 1$ and there are exactly $t-3$ numbers l_i equal $p_i - 1$. Therefore by Serre duality we get

$$\begin{aligned} \text{Ext}_{\mathbb{X}}^1(L(\vec{\omega}), L(\vec{x})) &\cong \text{DHom}_{\mathbb{X}}(L(\vec{x}), L(2\vec{\omega})) \cong S_{2\vec{\omega}-\vec{x}} = 0, \\ \text{Ext}_{\mathbb{X}}^1(L(\vec{x}), L(\vec{\omega})) &\cong \text{DHom}_{\mathbb{X}}(L(\vec{\omega}), L(\vec{x} + \vec{\omega})) \cong S_{\vec{x}} \cong k. \end{aligned}$$

Applying the functor $\text{Hom}_{\mathbb{X}}(L(\vec{x}), -)$ to η we get an exact sequence

$$\begin{aligned} 0 \longrightarrow \underbrace{\text{Hom}_{\mathbb{X}}(L(\vec{x}), L(\vec{\omega}))}_{=0} \longrightarrow \text{Hom}_{\mathbb{X}}(L(\vec{x}), E) \longrightarrow \underbrace{\text{Hom}_{\mathbb{X}}(L(\vec{x}), L(\vec{x}))}_{\cong k} \longrightarrow \\ \xrightarrow{\delta} \underbrace{\text{Ext}_{\mathbb{X}}^1(L(\vec{x}), L(\vec{\omega}))}_{\cong k} \longrightarrow \text{Ext}_{\mathbb{X}}^1(L(\vec{x}), E) \longrightarrow \underbrace{\text{Ext}_{\mathbb{X}}^1(L(\vec{x}), L(\vec{x}))}_{=0} \longrightarrow 0. \end{aligned}$$

Now, $\delta(1_{L(\vec{x})}) = \eta \neq 0$ because η does not split and consequently δ is isomorphism. Therefore

$$\text{Hom}_{\mathbb{X}}(L(\vec{x}), F) = 0 = \text{Ext}_{\mathbb{X}}^1(L(\vec{x}), F).$$

The long exact sequence $\text{Hom}_{\mathbb{X}}(\eta, L(\vec{\omega}))$ has the form

$$\begin{aligned} 0 \longrightarrow \underbrace{\text{Hom}_{\mathbb{X}}(L(\vec{x}), L(\vec{\omega}))}_{=0, \text{ because } \vec{\omega} - \vec{x} < 0} \xrightarrow{-\circ\pi} \text{Hom}_{\mathbb{X}}(E, L(\vec{\omega})) \xrightarrow{-\circ i} \underbrace{\text{Hom}_{\mathbb{X}}(L(\vec{\omega}), L(\vec{\omega}))}_{\cong k} \longrightarrow \\ \longrightarrow \underbrace{\text{Ext}_{\mathbb{X}}^1(L(\vec{x}), L(\vec{\omega}))}_{\cong k} \longrightarrow \text{Ext}_{\mathbb{X}}^1(E, L(\vec{\omega})) \longrightarrow \underbrace{\text{Ext}_{\mathbb{X}}^1(L(\vec{\omega}), L(\vec{\omega}))}_{=0} \longrightarrow 0. \end{aligned}$$

Let $u : E \rightarrow L(\vec{\omega})$ be a non-zero morphism. Then $u \circ i : L(\vec{\omega}) \rightarrow L(\vec{\omega})$ is the zero map. Indeed, if $u \circ i$ is non-zero, it is an isomorphism and so η splits which is impossible. Hence $u \in \ker(-\circ i) = \text{Im}(-\circ\pi) = 0$, because $\text{Hom}_{\mathbb{X}}(L(\vec{x}), L(\vec{\omega})) = 0$. Then $\text{Hom}_{\mathbb{X}}(E, L(\vec{\omega})) = 0$ and by comparing dimensions in the sequence $\text{Hom}_{\mathbb{X}}(\eta, L(\vec{\omega}))$ we obtain that $\text{Ext}_{\mathbb{X}}^1(E, L(\vec{\omega})) = 0$.

(iv) \Rightarrow (i). Assume that the vector spaces $\text{Ext}_{\mathbb{X}}^1(L(\vec{\omega}), L(\vec{x}))$, $\text{Hom}_{\mathbb{X}}(E, L(\vec{\omega}))$, $\text{Ext}_{\mathbb{X}}^1(E, L(\vec{\omega}))$, $\text{Hom}_{\mathbb{X}}(L(\vec{x}), E)$ and $\text{Ext}_{\mathbb{X}}^1(L(\vec{x}), E)$ vanish. We apply the functor $\text{Hom}_{\mathbb{X}}(L(\vec{\omega}), -)$ to η and obtain a long exact sequence

$$\begin{aligned} 0 \longrightarrow \underbrace{\text{Hom}_{\mathbb{X}}(L(\vec{\omega}), L(\vec{\omega}))}_{\cong k} \longrightarrow \text{Hom}_{\mathbb{X}}(L(\vec{\omega}), E) \longrightarrow \underbrace{\text{Hom}_{\mathbb{X}}(L(\vec{\omega}), L(\vec{x}))}_{=0} \longrightarrow \\ \longrightarrow \underbrace{\text{Ext}_{\mathbb{X}}^1(L(\vec{\omega}), L(\vec{\omega}))}_{=0} \longrightarrow \text{Ext}_{\mathbb{X}}^1(L(\vec{\omega}), E) \longrightarrow \underbrace{\text{Ext}_{\mathbb{X}}^1(L(\vec{\omega}), L(\vec{x}))}_{=0} \longrightarrow 0. \end{aligned}$$

Hence $\text{Hom}_{\mathbb{X}}(L(\vec{\omega}), E) \cong k$ and $\text{Ext}_{\mathbb{X}}^1(L(\vec{\omega}), E) = 0$. Finally, we apply the functor $\text{Hom}_{\mathbb{X}}(-, E)$ to η and obtain a long exact sequence

$$\begin{aligned} 0 \longrightarrow \underbrace{\text{Hom}_{\mathbb{X}}(L(\vec{x}), E)}_{=0} \longrightarrow \text{Hom}_{\mathbb{X}}(E, E) \longrightarrow \underbrace{\text{Hom}_{\mathbb{X}}(L(\vec{\omega}), E)}_{\cong k} \longrightarrow \\ \longrightarrow \underbrace{\text{Ext}_{\mathbb{X}}^1(L(\vec{x}), E)}_{=0} \longrightarrow \text{Ext}_{\mathbb{X}}^1(E, E) \longrightarrow \underbrace{\text{Ext}_{\mathbb{X}}^1(L(\vec{\omega}), E)}_{=0} \longrightarrow 0. \end{aligned}$$

Therefore $\text{Hom}_{\mathbb{X}}(E, E) = k$ and $\text{Ext}_{\mathbb{X}}^1(E, E) = 0$, so the vector bundle E is exceptional. \square

For an exceptional bundle E the non-split sequence η uniquely determines E , and in this case we will say that the *extension bundle* is given by the pair (L, \vec{x}) and we will denote it by $E_L\langle \vec{x} \rangle$.

Theorem 3. *Let E be an indecomposable vector bundle over \mathbb{X} such that there is a short exact sequence*

$$\eta : \quad 0 \longrightarrow L(\vec{\omega}) \xrightarrow{i} E \xrightarrow{\pi} L(\vec{x}) \longrightarrow 0,$$

where $\vec{x} = \sum_{i=1}^t l_i \vec{x}_i$, with $0 \leq l_i \leq p_i - 1$ and $0 \leq \vec{x} \leq \vec{\delta}$. Moreover, let $I = \{i \mid l_i \neq p_i - 1\}$. Then

$$\begin{aligned}\mathfrak{P}(E_L\langle\vec{x}\rangle) &= L(\vec{\omega}) \oplus \bigoplus_{j \in I} L(\vec{x} - (1 + l_j)\vec{x}_j) \\ \mathfrak{I}(E_L\langle\vec{x}\rangle) &= L(\vec{x}) \oplus \bigoplus_{j \in I} L((1 + l_j)\vec{x}_j + \vec{\omega})\end{aligned}$$

Further, the line bundle summands of $\mathfrak{P}(E_L\langle\vec{x}\rangle)$ (resp. $\mathfrak{I}(E_L\langle\vec{x}\rangle)$) are mutually $\text{Hom}_{\mathbb{X}}$ -orthogonal.

Proof. Observe that the condition $0 \leq \vec{x} \leq \vec{\delta}$ implies that $\#I \geq 3$. We will consider the case of injective hulls, the arguments for projective covers are dual.

From Serre duality we obtain that $\text{Ext}_{\mathbb{X}}^1(L(\vec{x}), L(\vec{\omega} + (1 + l_j)\vec{x}_j)) = 0$ for $j \in I$. Hence, applying the functor $\text{Hom}_{\mathbb{X}}(-, L(\vec{\omega} + (1 + l_j)\vec{x}_j))$ for $j \in I$ to η we see that there are morphisms $\widetilde{x_j^{l_j+1}} : E \rightarrow L(\vec{\omega} + (1 + l_j)\vec{x}_j)$ such that $\widetilde{x_j^{l_j+1}} \circ i = x_j^{l_j+1}$ where $x_j^{l_j+1} : L(\vec{\omega}) \rightarrow L(\vec{\omega} + (1 + l_j)\vec{x}_j)$. We will show that $j_E = \left(\pi, \left(\widetilde{x_j^{l_j+1}} \right)_{j \in I} \right) : E \rightarrow L(\vec{x}) \oplus \bigoplus_{j \in I} L(\vec{\omega} + (1 + l_j)\vec{x}_j)$ is an injective hull of the bundle E . For this we will prove that each morphism $E \rightarrow L'$ where L' is a line bundle, factors through $L(\vec{x}) \oplus \bigoplus_{j \in I} L(\vec{\omega} + (1 + l_j)\vec{x}_j)$. For simplicity we can write L' as $L(\vec{\omega} + \vec{z})$ for some $\vec{z} \in \mathbb{L}$. Remark, that for $j \notin I$ the space $\text{Ext}_{\mathbb{X}}^1(L(\vec{x}), L(\vec{\omega} + (1 + l_j)\vec{x}_j)) \neq 0$, and there are no maps $\widetilde{x_j^{l_j+1}}$ for $j \notin I$.

First, we show that

$$(\star) \quad \text{Hom}_{\mathbb{X}}(E, L(\vec{\omega} + \vec{z})) = 0 \quad \text{for } 0 \leq \vec{z} \leq \vec{x}.$$

Indeed, let $\vec{z} = \sum_{j=1}^t a_j \vec{x}_j$ be an element of \mathbb{L} with $0 \leq a_j \leq l_j$ for all j and let $z = x_1^{a_1} x_2^{a_2} \cdots x_t^{a_t}$ be a morphism from $L(\vec{\omega})$ to $L(\vec{\omega} + \vec{z})$. Applying the functor $\text{Hom}_{\mathbb{X}}(E, -)$ to the sequence $0 \rightarrow L(\vec{\omega}) \xrightarrow{z} L(\vec{\omega} + \vec{z}) \rightarrow S \rightarrow 0$ we obtain that $\text{Ext}_{\mathbb{X}}^1(E, L(\vec{\omega})) = 0$. Next, applying the functor $\text{Hom}_{\mathbb{X}}(-, L(\vec{\omega} + \vec{z}))$ to η we obtain a long exact sequence

$$\begin{aligned}0 \rightarrow \text{Hom}_{\mathbb{X}}(E, L(\vec{\omega} + \vec{z})) \rightarrow \underbrace{\text{Hom}_{\mathbb{X}}(L(\vec{\omega}), L(\vec{\omega} + \vec{z}))}_{\cong k} \rightarrow \\ \rightarrow \underbrace{\text{Ext}_{\mathbb{X}}^1(L(\vec{x}), L(\vec{\omega} + \vec{z}))}_{\cong k} \rightarrow \underbrace{\text{Ext}_{\mathbb{X}}^1(E, L(\vec{\omega} + \vec{z}))}_{= 0} \rightarrow \cdots\end{aligned}$$

By comparing dimensions we get $\text{Hom}_{\mathbb{X}}(E, L(\vec{\omega} + \vec{z})) = 0$.

Let $h : E \rightarrow L(\vec{\omega} + \vec{z})$ be a non-zero morphism for some $z \in \mathbb{L}$. Because η is a non-split exact sequence, the map $h \circ i : L(\vec{\omega}) \xrightarrow{i} E \xrightarrow{h} L(\vec{\omega} + \vec{z})$ is a non-isomorphism. If $h \circ i$ is the zero map, then h factors through by π and we are done. Suppose now that $h \circ i \neq 0$. Then $\vec{z} \geq 0$, because $\text{Hom}_{\mathbb{X}}(L(\vec{\omega}), L(\vec{\omega} + \vec{z})) \neq 0$. Moreover, from property (\star) we have $\vec{z} \not\leq \vec{x}$, and so $\vec{x} - \vec{z} \not\leq 0$. Now, we prove that there are maps $h_j : L((l_j + 1)\vec{x}_j + \vec{\omega}) \rightarrow L(\vec{\omega} + \vec{z})$ for $j \in I$ such that $h \circ i = \sum_{j \in I} h_j \circ x_j^{l_j+1}$. Since $z \geq 0$ and $\vec{x} - \vec{z} \not\leq 0$, after standard calculation in the group $\mathbb{L}(\underline{p})$, we see that there is an index $j_0 \in I$ such that $\vec{z} - (l_{j_0} + 1)\vec{x}_{j_0} \geq 0$. Therefore

there is a map $h_{j_0} : L((l_{j_0} + 1)\vec{x}_{j_0} + \vec{\omega}) \longrightarrow L(\vec{\omega} + \vec{z})$ such that $h \circ i = h_{j_0} \circ x_{j_0}^{l_{j_0} + 1}$. Further we define $h_j = 0$ for $j \neq j_0$.

Then we have

$$\left(h - \sum_{j \in I} h_j \circ \widetilde{x_j^{l_j + 1}} \right) \circ i = h \circ i - \sum_{j \in I} h_j \circ (\widetilde{x_j^{l_j + 1}} \circ i) = h \circ i - \sum_{j \in I} h_j \circ x_j^{l_j + 1} = 0,$$

and we conclude that $h - \sum_{j \in I} h_j \circ \widetilde{x_j^{l_j + 1}} \in \ker(- \circ i) = \text{Im}(- \circ \pi)$. Thus there is a map $g : L(\vec{x}) \longrightarrow L(\vec{\omega} + \vec{z})$ such that $g \circ \pi = h - \sum_{j \in I} h_j \circ \widetilde{x_j^{l_j + 1}}$ and hence $h = g \circ \pi + \sum_{j \in I} h_j \circ \widetilde{x_j^{l_j + 1}}$.

The $\text{Hom}_{\mathbb{X}}$ -orthogonality is easy to check. The minimality for the map $j_E : E \longrightarrow L(\vec{x}) \oplus \bigoplus_{j \in I} L((l_j + 1)\vec{x}_j + \vec{\omega})$ follows then from the $\text{Hom}_{\mathbb{X}}$ -orthogonality of the line bundles $L(\vec{x})$ and $L((l_j + 1)\vec{x}_j + \vec{\omega})$ for $j \in I$. \square

Remark 4.1. In [9] it was shown that in the case of weight type $(2, a, b)$ the suspension functor [1] in the stable vector bundle category $\underline{\text{vect}}(\mathbb{X})$ coincides with the shift functor by \vec{x}_1 . Therefore there is a short exact sequence

$$0 \longrightarrow \mathfrak{P}(E)(-\vec{x}_1) \longrightarrow E \longrightarrow \mathfrak{P}(E) \longrightarrow 0$$

for each indecomposable bundle E . Hence in this case we have $\text{rk}\mathfrak{P}(E) = 2\text{rk}E$. From the theorem above we see that in the case of t weights with $t > 3$ there is an indecomposable, not exceptional rank two bundle such that $\text{rk}\mathfrak{P}(E) > 4 = 2\text{rk}E$.

For example in the case $(2, 2, 2, 3)$ the projective cover of an indecomposable bundle of the data (L, \vec{x}_4) has rank 5. Therefore in the case $(2, p_2, p_3, \dots, p_t)$ and $t > 3$ the suspension functor cannot be realized by a shift with an element from \mathbb{L} .

In the same way as Lemma 3.1 and Lemma 3.2. the section 3, we can prove the following two lemmata.

Lemma 4.2. *Let E be a non-zero vector bundle over \mathbb{X} of a type (p_1, p_2, \dots, p_t) with a projective cover $\mathfrak{P}(E)$. If $\text{Ext}_{\mathbb{X}}^1(T, E) = 0$ for the canonical bundle T , then there is a line bundle L in the decomposition $\mathfrak{P}(E)$ into a direct sum of line bundles such that $\text{Ext}_{\mathbb{X}}^1(T, L) = 0$. \square*

From Theorem 2 each extension bundle can be given by a line bundle L and an element $\vec{x} = \sum_{i \in I} l_i \vec{x}_i + \sum_{i \notin I} (p_i - 1) \vec{x}_i \in \mathbb{L}$ for some $I \subset \{1, 2, \dots, t\}$ with $\#I = 3$.

Lemma 4.3. *Let \mathbb{X} be a weighted projective line of type (p_1, p_2, \dots, p_t) and let E be an extension bundle given by a pair (L, \vec{x}) , where $\vec{x} = \sum_{i \in I} l_i \vec{x}_i + \sum_{i \notin I} (p_i - 1) \vec{x}_i$. Denote by \widehat{L}_i the direct summand $L(\vec{x} - (1 + l_i) \vec{x}_i)$ of $\mathfrak{P}(E)$.*

(i) *There is an exact sequence*

$$0 \longrightarrow \widehat{L}_i \longrightarrow E \longrightarrow L((1 + l_j)x_j + \vec{\omega}) \longrightarrow 0,$$

where $L((1 + l_j)x_j + \vec{\omega})$ is a direct summand of $\mathfrak{P}(E)$.

(ii) *The extension bundle E can be determined by the following pairs*

$$\left(L(\vec{x} - (1 + l_i) \vec{x}_i)(-\vec{\omega}), \quad 2\vec{\omega} + 2(1 + l_i) \vec{x}_i - \vec{x} \right) \quad \text{for } i \in I.$$

(iii) *The element $2\vec{\omega} + 2(1 + l_i) \vec{x}_i - \vec{x}$ in normal form has exactly $t - 3$ coefficients equal to $p_i - 1$ for $i \notin I$.*

Proof. The statements (i) and (ii) are proved in the same way as in the case of three weights. The part (iii) follows from calculations in the group \mathbb{L} and is left to the reader. \square

The next result is a generalization of Theorem 1 and can be proved analogously using Lemma 4.2 and Lemma 4.3.

Theorem 4. *Let $\Lambda = \Lambda(\underline{p}, \underline{\lambda})$ be a canonical algebra with t arms. Then each exceptional Λ -module of rank two can be described by matrices having entries 0, λ_i and $-\lambda_i$. \square*

5. EXCEPTIONAL COKERNELS

In this section we will deal with cokernels of maps of the form

$$(1) \quad \left[x_i^{b_i} \right]_{i \in I} : \mathcal{O}(\vec{y}) \longrightarrow \bigoplus_{i \in I} \mathcal{O}(\vec{y} + b_i \vec{x}_i),$$

where $I \subset \{1, 2, \dots, t\}$, $0 < b_i < p_i - 1$ for $i \in I$ and $\vec{y} \in \mathbb{L}_+$. We will prove that such cokernels are exceptional modules. Moreover every exceptional module of rank two, can be obtain in this way. Finally, by the cokernel construction, we compute matrices for each exceptional module of rank two.

Lemma 5.1. *Consider an exact sequence*

$$(\star) \quad 0 \longrightarrow F \xrightarrow{f} G \xrightarrow{\pi} E \longrightarrow 0$$

in $\text{coh}(\mathbb{X})$ such that the following conditions are satisfied.

- C1. F is exceptional,
- C2. $\text{Hom}_{\mathbb{X}}(G, F) = 0 = \text{Ext}_{\mathbb{X}}^1(G, F)$,
- C3. $\text{Ext}_{\mathbb{X}}^1(G, G) = 0$,
- C4. The map $-\circ f : \text{End}_{\mathbb{X}}(G) \longrightarrow \text{Hom}_{\mathbb{X}}(F, G)$ is an isomorphism.

Then the following properties holds:

- (i) $\text{Ext}_{\mathbb{X}}^1(E, F) \cong k$ and $\text{Hom}_{\mathbb{X}}(E, G) = 0 = \text{Ext}_{\mathbb{X}}^1(E, G)$,
- (ii) E is exceptional,
- (iii) Up to an isomorphism E does not depend on the map f .

Proof. (i). Applying the functor $\text{Hom}_{\mathbb{X}}(-, F)$ to the exact sequence (\star) we obtain a long exact sequence

$$\cdots \longrightarrow \underbrace{\text{Hom}_{\mathbb{X}}(G, F)}_{=0} \longrightarrow \text{End}_{\mathbb{X}}(F) \longrightarrow \text{Ext}_{\mathbb{X}}^1(E, F) \longrightarrow \underbrace{\text{Ext}_{\mathbb{X}}^1(G, F)}_{=0} \longrightarrow \cdots$$

Therefore $\text{Ext}_{\mathbb{X}}^1(E, F) \cong \text{End}_{\mathbb{X}}(F) \cong k$.

Furthermore, applying the functor $\text{Hom}_{\mathbb{X}}(-, G)$ to (\star) we obtain a long exact sequence

$$\begin{aligned} 0 \longrightarrow \text{Hom}_{\mathbb{X}}(E, G) \xrightarrow{-\circ\pi} \text{End}_{\mathbb{X}}(G) \xrightarrow{-\circ f} \text{Hom}_{\mathbb{X}}(F, G) \xrightarrow{\delta} \text{Ext}_{\mathbb{X}}^1(E, G) \longrightarrow \\ \longrightarrow \underbrace{\text{Ext}_{\mathbb{X}}^1(G, G)}_{=0} \longrightarrow \text{Ext}_{\mathbb{X}}^1(F, G) \longrightarrow 0. \end{aligned}$$

Because $-\circ f$ is an isomorphism we get $\text{Hom}_{\mathbb{X}}(E, G) = 0$ and $\text{Ext}_{\mathbb{X}}^1(E, G) = 0$

(ii). Applying the functor $\mathrm{Hom}_{\mathbb{X}}(E, -)$ to the exact sequence (\star) we have a long exact sequence

$$\begin{aligned} 0 \longrightarrow \mathrm{Hom}_{\mathbb{X}}(E, F) \longrightarrow \overbrace{\mathrm{Hom}_{\mathbb{X}}(E, G)}^{=0} \longrightarrow \widehat{\mathrm{End}}_{\mathbb{X}}(E) \longrightarrow \\ \longrightarrow \mathrm{Ext}_{\mathbb{X}}^1(E, F) \longrightarrow \underbrace{\mathrm{Ext}_{\mathbb{X}}^1(E, G)}_{=0} \longrightarrow \mathrm{Ext}_{\mathbb{X}}^1(E, E) \longrightarrow 0. \end{aligned}$$

We conclude that $\mathrm{Ext}_{\mathbb{X}}^1(E, E) = 0$, and using (i) also that $\widehat{\mathrm{End}}_{\mathbb{X}}(E) \cong \mathrm{Ext}_{\mathbb{X}}^1(E, F) \cong k$.

(iii). Suppose that we have exact sequences

$$0 \longrightarrow F \xrightarrow{f} G \longrightarrow E \longrightarrow 0 \quad \text{and} \quad 0 \longrightarrow F \xrightarrow{\widehat{f}} G \longrightarrow \widehat{E} \longrightarrow 0$$

Then we have $[E] = [G] - [F] = [\widehat{E}]$ in the Grothendieck group $K_0(\mathbb{X})$. From (ii) we know that the sheaves E and \widehat{E} are exceptional. We infer from [12, Proposition 4.4.1] that $E \cong \widehat{E}$. \square

We will study the following cases of maps satisfying conditions C1.-C4. of the previous proposition.

- a.** A map $[x_1^{b_1}, \dots, x_t^{b_t}]^T : \mathcal{O} \longrightarrow \bigoplus_{i=1}^t \mathcal{O}(b_i \vec{x}_i)$, where $1 \leq b_i \leq p_i - 1$.
- b.** For $J \subseteq \{1, \dots, t\}$ we consider $[x_i^{b_i}]_{i \in J} : \mathcal{O} \longrightarrow \bigoplus_{i \in J} \mathcal{O}(b_i \vec{x}_i)$, where $1 \leq b_i \leq p_i - 1$ for $i = 1, \dots, t$.
- c.** If $f : F \longrightarrow G$ satisfies the conditions C1.-C4. then for each $\vec{x} \in \mathbb{L}$ the map $f(\vec{x}) : F(\vec{x}) \longrightarrow G(\vec{x})$ also satisfies these conditions.

Note that if G is in $\mathrm{mod}_+(\Lambda)$ in the sequence (\star) , then E is also in $\mathrm{mod}_+(\Lambda)$. Therefore in the cases **a.** and **b.** above the cokernels are exceptional Λ -modules.

Let $\mathbb{X} = \mathbb{X}(p_1, p_2, \dots, p_t)$ be a weighted projective line and let $b_i, i = 1, \dots, t$ are natural number such that $1 \leq b_i \leq p_i - 1$. Denote $I = \{i : b_i < p_i - 1\}$ and assume that $\#I = 3$.

Proposition 5.2. *The cokernel E of the exact sequence*

$$(\star) \quad 0 \longrightarrow L \xrightarrow{f = \begin{bmatrix} x_i^{b_i} \end{bmatrix}_{i \in I}} \bigoplus_{i \in I} L(b_i \vec{x}_i) \xrightarrow{\pi} E \longrightarrow 0$$

is an extension bundle with data

$$\left(L(b_{i_0} x_{i_0} - \vec{\omega}), \quad \vec{\omega} + \sum_{i \in I} b_i \vec{x}_i - 2b_{i_0} \vec{x}_{i_0} \right) \quad \text{for each } i_0 \in I.$$

Moreover for each $i_0 \in I$ the line bundle $L(b_{i_0} \vec{x}_{i_0})$ is a direct summand of the projective cover $\mathfrak{P}(E)$.

Proof. From Lemma 5.1 we conclude

$$\mathrm{Hom}_{\mathbb{X}}(E, L(b_{i_0} \vec{x}_{i_0})) = 0 = \mathrm{Ext}_{\mathbb{X}}^1(E, L(b_{i_0} \vec{x}_{i_0})) \quad \text{for each } i_0 \in I.$$

Applying the functor $\text{Hom}_{\mathbb{X}}(L(b_{i_0}\vec{x}_{i_0}), -)$ to (\star) we obtain a long exact sequence

$$\begin{aligned} 0 \longrightarrow \overbrace{\text{Hom}_{\mathbb{X}}(L(b_{i_0}\vec{x}_{i_0}), L)}^{=0} \longrightarrow \text{Hom}_{\mathbb{X}}\left(L(b_{i_0}\vec{x}_{i_0}), \bigoplus_{i=1}^3 L(b_i\vec{x}_i)\right) \longrightarrow \\ \longrightarrow \text{Hom}_{\mathbb{X}}(L(b_{i_0}\vec{x}_{i_0}), E) \longrightarrow \underbrace{\text{Ext}_{\mathbb{X}}^1(L(b_{i_0}\vec{x}_{i_0}), L)}_{=0} \longrightarrow \cdots \end{aligned}$$

Therefore $\text{Hom}_{\mathbb{X}}(L(b_{i_0}\vec{x}_{i_0}), E) \cong \text{Hom}_{\mathbb{X}}\left(L(b_{i_0}\vec{x}_{i_0}), \bigoplus_{i=1}^3 L(b_i\vec{x}_i)\right) \cong k$. We denote $\pi = [\pi_i]_{i \in I}$. Then each sequence

$$\eta_i : 0 \longrightarrow L(b_i\vec{x}_i) \xrightarrow{\pi_i} E \longrightarrow C_i \longrightarrow 0 \quad \text{for } i \in I$$

satisfies the conditions C1.-C3. from Lemma 5.1. Because $\text{End}_{\mathbb{X}}(E) \cong k \cong \text{Hom}_{\mathbb{X}}(L(b_i\vec{x}_i), E)$, the map $-\circ\pi_i : \text{End}_{\mathbb{X}}(E) \longrightarrow \text{Hom}_{\mathbb{X}}(L(b_i\vec{x}_i), E)$ is an isomorphism or is zero. Further, since $\pi \neq 0$, at least one map $-\circ\pi_i$ is an isomorphism.

Assume that $-\circ\pi_{j_0}$ is an isomorphism. Then from Lemma 5.1, the term C_{j_0} of η_{j_0} is exceptional, therefore it is a line bundle. Moreover $\det C_{j_0} = \det E - \det L(b_{j_0}\vec{x}_{j_0}) = \det L + \sum_{i \in I, i \neq j_0} b_i\vec{x}_i$, and it follows that $C_{j_0} \cong L\left(\sum_{i \in I, i \neq j_0} b_i\vec{x}_i\right)$. The exact sequence η_{j_0} can be written in the following form

$$0 \longrightarrow L(b_{j_0}\vec{x}_{j_0} - \vec{\omega})(\vec{\omega}) \longrightarrow E \longrightarrow L(b_{j_0}\vec{x}_{j_0} - \vec{\omega})(\vec{x}) \longrightarrow 0,$$

where

$$\begin{aligned} \vec{x} &= \vec{\omega} - b_{j_0}\vec{x}_{j_0} + \sum_{i \in I, i \neq j_0} b_i\vec{x}_i \\ &= (p_{j_0} - b_{j_0} - 1)\vec{x}_{j_0} + \sum_{i \in I, i \neq j_0} (b_i - 1)\vec{x}_i + \sum_{j \notin I} (p_j - 1)\vec{x}_j. \end{aligned}$$

Then the element \vec{x} satisfies the inequality $0 \leq \vec{x} \leq \vec{\delta}$, and consequently E is an extension bundle with the data $(L(b_{j_0}\vec{x}_{j_0} - \vec{\omega}), \vec{x})$.

Moreover, from Theorem 3, the line bundles $L(b_i\vec{x}_i)$ for $i \in I, i \neq j$ are direct summands of the projective cover of the vector bundle E . Thus, from Lemma 4.3 the bundle E is an extension bundle with the data $(L(b_{i_0}\vec{x}_{i_0} - \vec{\omega}), \vec{\omega} + \sum_{i \in I} b_i\vec{x}_i - 2b_{i_0}\vec{x}_{i_0})$ for each $i_0 \in I$. \square

Proposition 5.3. *Let $E = E_L(\vec{x})$ be an extension bundle with $\vec{x} = \sum_{i \in I} l_i\vec{x}_i + \sum_{j \notin I} (p_j - 1)\vec{x}_j$ and $\#I = 3$. Then E is the cokernel in the following exact sequence*

$$0 \longrightarrow L(\vec{x} - \vec{c}) \xrightarrow{\left[x_i^{p_i - l_i - 1}\right]_{i \in I}} \bigoplus_{i \in I} L(\vec{x} - (1 + l_i)\vec{x}_i) \longrightarrow E \longrightarrow 0.$$

Moreover, the line bundles $L(\vec{x} - (1 + l_i)\vec{x}_i)$ are direct summands of $\mathfrak{P}(E)$ and $L(\vec{x} - \vec{c})$ is direct summand of $\mathfrak{J}(E)(-\vec{c})$.

Proof. Consider the map $\left[x_i^{p_i - l_i - 1}\right]_{i \in I} : L(\vec{x} - \vec{c}) \longrightarrow \bigoplus_{i \in I} L(\vec{x} - (1 + l_i)\vec{x}_i)$. This map is a monomorphism and from the Proposition 5.2 the cokernel of the map $\left[x_i^{p_i - l_i - 1}\right]_{i \in I}$ is the extension bundle with data (L, \vec{x}) . Hence $\text{coker} \left[x_i^{p_i - l_i - 1}\right]_{i \in I} \cong$

$E_L\langle\vec{x}\rangle$. The second claim follows from the form of the projective cover and the injective hull of the extension bundle $E_L\langle\vec{x}\rangle$. \square

Lemma 5.4. *A line bundle L is in $\text{mod}_+(\Lambda)$ if and only if $\det L \geq 0$.*

Proof. Assume first that L is in $\text{mod}_+(\Lambda)$, ie. $\text{Ext}_{\mathbb{X}}^1(T, L) = 0$. Then $\text{Hom}_{\mathbb{X}}(T, L) \neq 0$, so there is an element $\vec{x} \in \mathbb{L}$ such that $0 \leq \vec{x} \leq \vec{c}$ and $\text{Hom}_{\mathbb{X}}(\mathcal{O}(\vec{x}), L) \neq 0$. Therefore $\det L - \vec{x} \geq 0$, so $\det L = \underbrace{\det L - \vec{x}}_{\geq 0} + \underbrace{\vec{x}}_{\geq 0} \geq 0$.

Now, assume that $\det L = n\vec{c} + \sum_{i=1}^t a_i \vec{x}_i \geq 0$. Let $\vec{x} \in \mathbb{L}$ satisfy that $0 \leq \vec{x} \leq \vec{c}$. Then $\vec{x} + \vec{\omega} - \det L = \vec{x} + \sum_{i=1}^t (p_i - a_i - 1)\vec{x}_i - (n+2)\vec{c} \not\geq 0$. Therefore $\text{Ext}_{\mathbb{X}}^1(\mathcal{O}(\vec{x}), L) \cong D\text{Hom}_{\mathbb{X}}(L, \mathcal{O}(\vec{x} + \vec{\omega})) = 0$, and consequently the line bundle L is in $\text{mod}_+(\Lambda)$. \square

Proposition 5.5. *Let E be an extension bundle.*

(i) *For each direct summand \widehat{L} of $\mathfrak{J}(E)$ there is a short exact sequence*

$$\eta_{\widehat{L}} : 0 \longrightarrow \widehat{L}(-\vec{c}) \longrightarrow \bigoplus_{i=1}^3 L_i \longrightarrow E \longrightarrow 0,$$

where the L_i are pairwise distinct direct summands of projective cover $\mathfrak{P}(E)$.

(ii) *If E is a Λ -module from $\text{mod}_+(\Lambda)$, then for at least one direct summand \widehat{L} of $\mathfrak{J}(E)$ the line bundle $\widehat{L}(-\vec{c})$ is in $\text{mod}_+(\Lambda)$ and $\eta_{\widehat{L}}$ is a sequence of Λ -modules.*

Proof. The statement (i) is a consequence of Lemma 4.3 and Proposition 5.3.

(ii). Since E is an extension bundle there is an exact sequence

$$0 \longrightarrow L(\vec{\omega}) \longrightarrow E \longrightarrow L(\vec{x}) \longrightarrow 0,$$

of Λ -modules, where $\vec{x} = \sum_{i \in I} l_i \vec{x}_i + \sum_{j \notin I} (p_j - 1)\vec{x}_j$ with $\#I = 3$ and $0 \leq l_i \leq p_i - 2$. Recall from Lemma 5.4 that $\det L(\vec{\omega}) \geq 0$ and $\det L(\vec{x}) \geq 0$.

The direct summands of the injective hull $\mathfrak{J}(E)$ are as follows

$$L(\vec{x}), \quad L(\vec{\omega} + (1 + l_i)\vec{x}_i) \quad \text{for } i \in I.$$

If the line bundle $L(\vec{x} - \vec{c})$ is in $\text{mod}_+(\Lambda)$, we put $\widehat{L} = L(\vec{x})$ and the claim holds. Assume now that $L(\vec{x} - \vec{c})$ does not belong to $\text{mod}_+(\Lambda)$, then $\det L(\vec{x} - \vec{c}) \not\geq 0$. We write $\det L$ in normal form $\det L = n\vec{c} + \sum_{i=1}^t a_i \vec{x}_i$ and define two numbers m and m_I as follows

$$m := \#\{i \mid i \notin I \wedge a_i > 0\} \quad m_I := \#\{i \mid i \in I \wedge a_i + l_i > p_i\}.$$

Then we can write $\det L(\vec{x})$ in normal form

$$\begin{aligned} \det L(\vec{x}) &= n\vec{c} + \sum_{i=1}^t a_i \vec{x}_i + \sum_{i \in I} l_i \vec{x}_i + \sum_{i \notin I} (p_i - 1)\vec{x}_i \\ &= n\vec{c} + \sum_{i \in I} (a_i + l_i)\vec{x}_i + \sum_{i \notin I} (p_i - 1 + l_i)\vec{x}_i \\ &= (n + m + m_I)\vec{c} + \sum_{i \in I} b_i \vec{x}_i + \sum_{i \notin I} c_i \vec{x}_i \geq 0, \end{aligned}$$

where

$$b_i = \begin{cases} a_i + l_i - p_i & \text{if } a_i + l_i \geq p_i \\ a_i + l_i & \text{if } a_i + l_i < p_i \end{cases} \quad \text{and} \quad c_i = \begin{cases} p_i - 1 & \text{if } l_i = 0 \\ l_i - 1 & \text{if } l_i > 0 \end{cases}$$

Since

$$\det L(\vec{x} - \vec{c}) = (n + m + m_I - 1)\vec{c} + \sum_{i \in I} b_i \vec{x}_i + \sum_{i \notin I} c_i \vec{x}_i \not\equiv 0,$$

we have $n + m + m_I = 0$, hence (\star) $n + m = -m_I$. Similarly we compute the determinant for the line bundle $L(\vec{\omega})$. We have

$$\det L(\vec{\omega}) = n\vec{c} + \sum_{i=1}^t a_i \vec{x}_i + (t-2)\vec{c} - \sum_{i=1}^t \vec{x}_i = (n+t-2)\vec{c} + \sum_{i=1}^t (a_i-1)\vec{x}_i,$$

where $\sum_{i=1}^t (a_i-1)\vec{x}_i = -(t-3-m)\vec{c} + \sum_{i \in I} (a_i-1)\vec{x}_i + \sum_{i \notin I} b_i \vec{x}_i$. We denote by d_i the number $p_i - 1$ if $a_i = 0$ or $a_i - 1$ if $a_i > 0$. Then

$$\begin{aligned} \det L(\vec{\omega}) &= (n+m+1)\vec{c} + \sum_{i \in I} (a_i-1)\vec{x}_i + \sum_{i \notin I} d_i \vec{x}_i \\ &= (1-m_I)\vec{c} + \sum_{i \in I} (a_i-1)\vec{x}_i + \sum_{i \notin I} d_i \vec{x}_i \geq 0. \end{aligned}$$

Therefore m_I is equal to 0 or 1. Moreover if $m_I = 1$ then $a_i > 0$ for all $i \in I$, and if $m_I = 0$, then at most one of the numbers a_i for $i \in I$ is 0.

In the case $m_I = 1$ there is an index $i_0 \in I$ such that $a_{i_0} + l_{i_0} \geq p_{i_0}$. Then

$$\begin{aligned} \det L(\vec{\omega} + (1+l_{i_0})\vec{x}_{i_0} - \vec{c}) &= -\vec{c} + \sum_{i \in I} (a_i-1)\vec{x}_i + (1+l_{i_0})\vec{x}_{i_0} + \sum_{i \notin I} d_i \vec{x}_i \\ &= (a_{i_0} + l_{i_0} - p_{i_0})\vec{x}_{i_0} + \sum_{i \in I, i \neq i_0} \underbrace{(a_i-1)}_{\geq 0} \vec{x}_i + \sum_{i \notin I} d_i \vec{x}_i \geq 0. \end{aligned}$$

Therefore $L(\vec{\omega} + (1+l_{i_0})\vec{x}_{i_0} - \vec{c})$ is in $\text{mod}_+(\Lambda)$ and we put $\widehat{L} = L(\vec{\omega} + (1+l_{i_0})\vec{x}_{i_0})$.

In the case that $m_I = 0$ if $a_i > 0$ for all $i \in I$ then each line bundle $L(\vec{\omega} + (1+l_i)\vec{x}_i - \vec{c})$ is in $\text{mod}_+(\Lambda)$ and each of those line bundles gives us the claim. If $a_{i_0} = 0$ for some $i_0 \in I$, then only $L(\vec{\omega} + (1+l_{i_0})\vec{x}_{i_0} - \vec{c})$ is in $\text{mod}_+(\Lambda)$ and we put $\widehat{L} = L(\vec{\omega} + (1+l_{i_0})\vec{x}_{i_0})$. □

As a conclusion of the previous proposition, we obtain an improvement of Lemma 3.1.

Corollary 1. *If an extension bundle E is a Λ -module, then at least three direct summands of $\mathfrak{B}(E)$ are also Λ -modules.* □

Recall that we work with a map of the form

$$f_{\vec{y}}^{b_{i_1}, b_{i_2}, b_{i_3}} = \left[x_i^{b_i} \right]_{i \in I} : \mathcal{O}(\vec{y}) \longrightarrow \bigoplus_{i \in I} \mathcal{O}(\vec{y} + b_i \vec{x}_i),$$

where $I = \{i_1, i_2, i_3 \mid i_1 < i_2 < i_3\}$ is a subset of $\{1, 2, \dots, t\}$, $0 < b_i < p_i - 1$ for $i \in I$ and $\vec{y} \in \mathbb{L}_+$. If we write the element \vec{y} in normal form

$$\vec{y} = n\vec{c} + \sum_{i=1}^t a_i \vec{x}_i, \quad n \in \mathbb{Z}_{\geq 0} \quad \text{and} \quad 0 \leq a_i \leq p_i - 1 \quad \text{for} \quad i = 1, 2, \dots, t,$$

then we distinguish the following 8 cases:

- A** $a_j + b_j < p_j$ for all $j \in I$,
- B₁** $a_{i_1} + b_{i_1} \geq p_{i_1}$ and $a_j + b_j < p_j$ for $j \in I - \{i_1\}$,
- B₂** $a_{i_2} + b_{i_2} \geq p_{i_2}$ and $a_j + b_j < p_j$ for $j \in I - \{i_2\}$,
- B₃** $a_{i_3} + b_{i_3} \geq p_{i_3}$ and $a_j + b_j < p_j$ for $j \in I - \{i_3\}$,
- C₁** $a_{i_1} + b_{i_1} < p_{i_1}$ and $a_j + b_j \geq p_j$ for $j \in I - \{i_1\}$,
- C₂** $a_{i_2} + b_{i_2} < p_{i_2}$ and $a_j + b_j \geq p_j$ for $j \in I - \{i_2\}$,
- C₃** $a_{i_3} + b_{i_3} < p_{i_3}$ and $a_j + b_j \geq p_j$ for $j \in I - \{i_3\}$,
- D** $a_i + b_i \geq p_i$ for all $i \in I$.

In the following lemma we proof that it is sufficient to study the cases **A** or **B₃**.

Lemma 5.6. *Each extension module can be obtained as a cokernel of the map $f_{\vec{y}}^{b_{i_1}, b_{i_2}, b_{i_3}}$ in the case **A** or **B₃**.*

Proof. We proof that the cokernels in the cases **B₁**, **B₂** and **D** are isomorphic to cokernels of the case **B₃**. Moreover the cokernels in the cases **C₁**, **C₂** and **C₃** are isomorphic to cokernels of the case **A**.

Let E be an extension module in the case **B₂**, thus E is the cokernel of a map

$$f_{\vec{y}}^{b_{i_1}, b_{i_2}, b_{i_3}}, \quad \text{where } a_{i_1} + b_{i_1} < p_{i_1}, \quad a_{i_2} + b_{i_2} \geq p_{i_2}, \quad a_{i_3} + b_{i_3} < p_{i_3}.$$

From Lemma 5.2, applied to i_1 , we infer that E is an extension bundle $E_L(\vec{x})$ with data (L, \vec{x}) such that

$$\det L = \vec{y} + b_{i_3} \vec{x}_{i_3} - \vec{\omega} = n\vec{c} + (a_{i_3} + b_{i_3}) \vec{x}_{i_3} + \sum_{j \neq i_3} a_j \vec{x}_j - \vec{\omega},$$

$$\vec{x} = \vec{\omega} + b_{i_1} \vec{x}_1 + b_{i_2} \vec{x}_{i_2} - b_{i_3} \vec{x}_{i_3},$$

Consider the map

$$f_{\vec{z}}^{d_{i_1}, d_{i_2}, d_{i_3}} = \begin{bmatrix} d_{i_1} \\ x_{i_1}^{d_{i_1}} \\ d_{i_2} \\ x_{i_2}^{d_{i_2}} \\ d_{i_3} \\ x_{i_3}^{d_{i_3}} \end{bmatrix} : \mathcal{O}(\vec{z}) \longrightarrow \bigoplus_{i=1}^3 \mathcal{O}(\vec{z} + b_i \vec{x}_i),$$

where

$$d_{i_1} = b_{i_1}, \quad d_{i_2} = p_{i_2} - b_{i_2}, \quad d_{i_3} = p_{i_3} - b_{i_3}$$

and

$$\begin{aligned} \vec{z} &= \vec{y} + b_{i_2} \vec{x}_{i_2} + b_{i_3} \vec{x}_{i_3} - \vec{c} = \\ &= n\vec{c} + (a_{i_2} + b_{i_2} - p_{i_2}) \vec{x}_{i_2} + (a_{i_3} + b_{i_3}) \vec{x}_{i_3} + \sum_{j \neq i_2, i_3} a_j \vec{x}_j. \end{aligned}$$

Then from Lemma 5.2, applied to i_2 , we conclude that the cokernel of the map $f_{\vec{z}}^{d_{i_1}, d_{i_3}, d_{i_3}}$ is an extension bundle with data $(\widehat{L}, \widehat{x})$, such that

$$\det \widehat{L} = \vec{z} + d_{i_2} \vec{x}_{i_2} - \vec{\omega} = n\vec{c} + (a_{i_3} + b_{i_3}) \vec{x}_{i_3} + \sum_{j \neq i_3} a_j \vec{x}_j - \vec{\omega},$$

$$\begin{aligned} \widehat{x} &= \vec{\omega} + d_{i_1} \vec{x}_{i_1} - d_{i_2} \vec{x}_{i_2} + d_{i_3} \vec{x}_{i_3} = \vec{\omega} + b_{i_1} \vec{x}_{i_1} - (p_{i_2} - b_{i_2}) \vec{x}_{i_2} + (p_{i_3} - b_{i_3}) \vec{x}_{i_3} = \\ &= \vec{\omega} + b_{i_1} \vec{x}_{i_1} + b_{i_2} \vec{x}_{i_2} - b_{i_3} \vec{x}_{i_3}. \end{aligned}$$

Therefore, the cokernels of the maps $f_{\vec{y}}^{b_{i_1}, b_{i_3}, b_{i_3}}$ and $f_{\vec{z}}^{d_{i_1}, d_{i_3}, d_{i_3}}$ are isomorphic. Furthermore we have that

$$\begin{aligned} d_{i_1} + a_{i_1} &= a_{i_1} + b_{i_1} < p_{i_1}, & d_{i_2} + (a_{i_2} + b_{i_2} - p_{i_2}) &= a_{i_2} < p_{i_2}, \\ d_{i_3} + (a_{i_3} + b_{i_3}) &= a_{i_3} + p_{i_3} \geq p_{i_3}, \end{aligned}$$

hence E is isomorphic to an extension module in the case of \mathbf{B}_3 .

For the other cases, we use the same kind of arguments. We only put in a table the choice of d_{i_1} , d_{i_2} , d_{i_3} and \vec{z} . For simplicity, in the case $a_i + b_i \geq p_i$, we denote $a_i + b_i - p_i$ by c_i .

Case	d_{i_1}	d_{i_2}	d_{i_3}	\vec{z}
\mathbf{B}_1	$p_{i_1} - b_{i_1}$	b_{i_2}	$p_{i_3} - b_{i_3}$	$n\vec{c} + c_{i_1}\vec{x}_{i_1} + (a_{i_3} + b_{i_3})\vec{x}_{i_3} + \sum_{j \neq i_1, i_3} a_j \vec{x}_j$
\mathbf{C}_1	b_{i_1}	$p_{i_2} - b_{i_2}$	$p_{i_3} - b_{i_3}$	$(n+1)\vec{c} + c_{i_2}\vec{x}_{i_2} + c_{i_3}\vec{x}_{i_3} + \sum_{j \neq i_2, i_3} a_j \vec{x}_j$
\mathbf{C}_2	$p_{i_1} - b_{i_1}$	b_{i_2}	$p_{i_3} - b_{i_3}$	$(n+1)\vec{c} + c_{i_1}\vec{x}_{i_1} + c_{i_3}\vec{x}_{i_3} + \sum_{j \neq i_1, i_3} a_j \vec{x}_j$
\mathbf{C}_3	$p_{i_1} - b_{i_1}$	$p_{i_2} - b_{i_2}$	b_{i_3}	$(n+1)\vec{c} + c_{i_1}\vec{x}_{i_1} + c_{i_2}\vec{x}_{i_2} + \sum_{j \neq i_1, i_2} a_j \vec{x}_j$
\mathbf{D}	$p_{i_1} - b_{i_1}$	b_{i_2}	$p_{i_3} - b_{i_3}$	$(n+1)\vec{c} + c_{i_1}\vec{x}_{i_1} + c_{i_3}\vec{x}_{i_3} + \sum_{j \neq i_1, i_3} a_j \vec{x}_j$

□

5.1. The cokernel construction. We consider an exact sequence of Λ -modules

$$0 \longrightarrow L \xrightarrow{f} M \xrightarrow{g} N \longrightarrow 0,$$

and we assume that representations

$$L = \left(\{L_{\vec{x}}\}_{\vec{x} \in Q_0}, \{L_{\alpha}\}_{\alpha \in Q_1} \right), \quad M = \left(\{M_{\vec{x}}\}_{\vec{x} \in Q_0}, \{M_{\alpha}\}_{\alpha \in Q_1} \right)$$

by vector spaces and matrices and also the morphism $f = (f_{\vec{x}})_{\vec{x} \in Q_0}$ are known. Here Q_0 and Q_1 denote the set of vertices and arrows of the quiver of the canonical algebra, respectively.

We will construct a representation $\left(\{N_{\vec{x}}\}_{\vec{x} \in Q_0}, \{N_{\alpha}\}_{\alpha \in Q_1} \right)$ for N . The vector space $N_{\vec{x}}$ is the cokernel of the linear map $f_{\vec{x}}$, and $g_{\vec{x}}$ the reduction modulo $\text{Im} f_{\vec{x}}$. Let v_1, \dots, v_m be a basis of $M_{\vec{x}}$ and let $\dim_k \text{Im} f_{\vec{x}} = l$. We have that $M_{\vec{x}} = \text{Im} f_{\vec{x}} \oplus kv_1 \oplus \dots \oplus kv_{m-l}$ and that the set $v_1 + \text{Im} f_{\vec{x}}, \dots, v_{m-l} + \text{Im} f_{\vec{x}}$ is a basis of the linear space $N_{\vec{x}} = M_{\vec{x}} / \text{Im} f_{\vec{x}}$. Moreover, for $j \in \{1, \dots, l\}$, we have $v_{m-l+j} = f_{\vec{x}}(w_j) + a_{1,j}v_1 + \dots + a_{m-l,j}v_{m-l}$ for some $a_{i,j} \in k$ i $w_j \in L_{\vec{x}}$. Then

$$\begin{aligned} v_i &\xrightarrow{g_{\vec{x}}} v_i + \text{Im} f_{\vec{x}}, & \text{for } i = 1, \dots, m-l \\ v_{m-l+j} &\xrightarrow{g_{\vec{x}}} \sum_{i=1}^{m-l} a_{i,j} v_i + \text{Im} f_{\vec{x}}, & \text{for } i = 1, \dots, m-l. \end{aligned}$$

Therefore $g_{\vec{x}} = \begin{array}{|c|c|} \hline I_{m-l} & A \\ \hline \end{array}$, where $A = [a_{i,j}] \in M_{m-l,l}(k)$.

Next we will determine the maps N_α for $\alpha \in Q_1$. Let $\alpha : \vec{x} \rightarrow \vec{y}$ be an arrow of the quiver of the algebra Λ . Then the following diagram

$$(2) \quad \begin{array}{ccccccccc} 0 & \longrightarrow & L_{\vec{y}} & \xrightarrow{f_{\vec{y}}} & M_{\vec{y}} & \xrightarrow{g_{\vec{y}}} & N_{\vec{y}} & \longrightarrow & 0 \\ & & \downarrow L_\alpha & & \downarrow M_\alpha & & & & \\ 0 & \longrightarrow & L_{\vec{x}} & \xrightarrow{f_{\vec{x}}} & M_{\vec{x}} & \xrightarrow{g_{\vec{x}}} & N_{\vec{x}} & \longrightarrow & 0 \end{array}$$

can be uniquely completed to a commutative diagram by the map $N_\alpha : N_{\vec{y}} \rightarrow N_{\vec{x}}$, $v + \text{Im} f_{\vec{y}} \mapsto M_\alpha(v) + \text{Im} f_{\vec{x}}$.

It is easily checked that the maps N_α satisfy the canonical relations.

5.2. Construction of modules of type A. Let $I \subset \{1, 2, \dots, t\}$ with $\#I = 3$, and let $\underline{b} = (b_i)_{i \in I}$ with $1 \leq b_i \leq p_i - 1$. For I and \underline{b} we consider an exact sequence of vector bundles

$$0 \longrightarrow L \xrightarrow{\begin{bmatrix} x_i^{b_i} \end{bmatrix}_{i \in I}} \bigoplus_{i \in I} L(b_i \vec{x}_i) \xrightarrow{g} \text{coker} \begin{bmatrix} x_i^{b_i} \end{bmatrix}_{i \in I} =: E \longrightarrow 0,$$

where L is a line bundle, with $\det L = n\vec{c} + \sum_{i \in I} a_i \vec{x}_i \geq 0$ and $a_i + b_i < p_i$ for $i \in I$.

We denote by I_n the identity matrix of size n and by $X_{n+m \times n}$, $Y_{n+m \times n}$, $Z_n(\lambda)$ the following matrices

$$X_{n+m \times n} := \begin{bmatrix} I_n \\ 0 \end{bmatrix} \in M_{n+m, n}(k), \quad Y_{n+m \times n} := \begin{bmatrix} 0 \\ I_n \end{bmatrix} \in M_{n+m, n}(k),$$

$$Z_n(\lambda) := \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \\ \lambda & 1 & & 0 & 0 \\ & & \ddots & \ddots & \\ 0 & 0 & \cdots & \lambda & 1 \end{bmatrix} \in M_n(k).$$

The Λ -module attached to the line bundle L has the following shape:

$$L : \quad \begin{array}{ccccccc} & & k^{n+1} & \xleftarrow{\mathbb{1}} \cdots \xleftarrow{\mathbb{1}} & k^{n+1} & \xleftarrow{L_{\alpha_{a_1+1}}^{(1)} = X_{n+1 \times n}} & k^n & \xleftarrow{\mathbb{1}} \cdots \xleftarrow{\mathbb{1}} & k^n \\ & \swarrow & & & & & & & & \searrow \\ & & k^{n+1} & \xleftarrow{\mathbb{1}} \cdots \xleftarrow{\mathbb{1}} & k^{n+1} & \xleftarrow{L_{\alpha_{a_2+1}}^{(2)} = Y_{n+1 \times n}} & k^n & \xleftarrow{\mathbb{1}} \cdots \xleftarrow{\mathbb{1}} & k^n & \\ & \swarrow & & & & & & & & \searrow \\ k^{n+1} & \xleftarrow{\mathbb{1}} & \cdots & & \cdots & & \cdots & \xleftarrow{\mathbb{1}} & k^n & \\ & \swarrow & & & & & & & & \searrow \\ & & k^{n+1} & \xleftarrow{\mathbb{1}} \cdots \xleftarrow{\mathbb{1}} & k^{n+1} & \xleftarrow{L_{\alpha_{a_i+1}}^{(i)} = Y_{n+1 \times n}} & k^n & \xleftarrow{\mathbb{1}} \cdots \xleftarrow{\mathbb{1}} & k^n & \\ & \swarrow & & & & & & & & \searrow \\ Z_{n+1}(\lambda_t) & \xleftarrow{Z_{n+1}(\lambda_t)} & k^{n+1} & \xleftarrow{\mathbb{1}} \cdots \xleftarrow{\mathbb{1}} & k^{n+1} & \xleftarrow{L_{\alpha_{a_3+1}}^{(t)} = X_{n+1 \times n}} & k^n & \xleftarrow{\mathbb{1}} \cdots \xleftarrow{\mathbb{1}} & k^n & \end{array}$$

where $\mathbb{1}$ is the identity map (see [13, Proposition 3.4]). If $a_i = 0$ for some $i \geq 3$, then $L_{\alpha_1^{(i)}} = L_{\alpha_{a_i+1}^{(i)}} = Z_{n+1}(\lambda_i) \cdot X_{n+1 \times n}$.

The modules $L(b_i \vec{x}_i)$ for $i \in I$ have a similar shape, with the difference that in the i -th arm, the jump of dimension is realized for the arrow $\alpha_{a_i+b_i+1}^{(i)}$.

First we compute matrices of maps $x_i^{b_i} : L \rightarrow L(b_i \vec{x}_i)$ for each $i \in I$. The map $x_i^{b_i}$ has the following matrices:

$$\mu_i \cdot \left(\begin{array}{ccccccc} & \cdots & & \boxed{I_{n+1}}^{a_1 \vec{x}_1} & \boxed{I_n}^{(a_1+1) \vec{x}_1} & \cdots & \\ & & & & & & \\ \cdots & \boxed{I_{n+1}}^{a_i \vec{x}_i} & \boxed{X_{n+1 \times n}}^{(a_i+1) \vec{x}_i} & \cdots & \boxed{X_{n+1 \times n}}^{(a_i+b_i) \vec{x}_i} & \boxed{I_n}^{(a_i+b_i+1) \vec{x}_i} & \cdots \\ \boxed{I_{n+1}} & & & \cdots & & & \boxed{I_n} \\ & \cdots & & \boxed{I_{n+1}}^{a_j \vec{x}_j} & \boxed{I_n}^{(a_j+1) \vec{x}_j} & \cdots & \\ & & & \cdots & & & \end{array} \right),$$

for some $\mu_i \in k$, where $i \neq 2$ and $j \neq i$. Here the captions above the frames mean the vertices of the quiver and the matrices in the frames the matrices for them. Moreover, in the case $i = 2$ we need to switch from $X_{n+1 \times n}$ to $Y_{n+1 \times n}$. Therefore the map $[x_i^{b_i}]_{i \in I}$ depends on the three scalars μ_{i_1} , μ_{i_2} and μ_{i_3} . From Lemma 5.1 (iii), we can put $\mu_{i_1} = 1 = \mu_{i_2}$ and $\mu_{i_3} = -1$.

The second step is the computation of the map $g : \bigoplus_{i \in I} L(b_i \vec{x}_i) \rightarrow E$. For this purpose we will use the following lemma from linear algebra, where for simplicity, we will use notation $B_{b \times a}$ for a matrix $B \in M_{b \times a}(k)$.

Lemma 5.7. *Let $0 \rightarrow V \xrightarrow{f} W \xrightarrow{-} \text{coker}(f) \rightarrow 0$ be an exact sequence of linear maps, where $\dim V = a$, $\dim W = a + b + c$ and $-$ is the reduction modulo $\text{Im}(f)$.*

(1) If f has a block matrix form $\begin{array}{|c|} \hline B_{b \times a} \\ \hline C_{c \times a} \\ \hline -I_a \\ \hline \end{array}$, then the reduction map $-$ has a block matrix form

$$\begin{array}{|c|c|c|} \hline I_b & 0 & B_{b \times a} \\ \hline 0 & I_c & C_{c \times a} \\ \hline \end{array}.$$

(2) If f has a block matrix form $\begin{array}{|c|} \hline I_a \\ \hline B_{b \times a} \\ \hline -C_{c \times a} \\ \hline \end{array}$, then the reduction map $-$ has a block matrix form

$$\begin{array}{|c|c|c|} \hline -B_{b \times a} & I_b & 0 \\ \hline C_{c \times a} & 0 & I_c \\ \hline \end{array}.$$

Proof. (i) Let v_1, \dots, v_a be a basis of V , and let $w_1^1, \dots, w_b^1, w_1^2, \dots, w_c^2, w_1^3, \dots, w_a^3$ be a basis of W . Furthermore, we choose $\overline{w_1^1}, \dots, \overline{w_b^1}, \overline{w_1^2}, \dots, \overline{w_c^2}$ as a basis of $\text{coker}(f)$. Then the equalities

$$\overline{w_i^3} = \overline{w_i^3 + f(w_i^3)} = \overline{Bw_i^3 + Cw_i^3}, \quad \text{for } i = 1, 2, \dots, a$$

implies the claim.

(ii) We choose v_1, \dots, v_a as a basis of V , and $w_1^1, \dots, w_a^1, w_1^2, \dots, w_b^2, w_1^3, \dots, w_c^3$ as a basis of W . Further, we choose $\overline{w_1^2}, \dots, \overline{w_b^2}, \overline{w_1^3}, \dots, \overline{w_c^3}$ as a basis of $\text{coker}(f)$. Then

$$\overline{w_i^1} = \overline{w_i^1 - f(w_i^1)} = \overline{-Bw_i^1 + Cw_i^1}, \quad \text{for } i = 1, 2, \dots, a$$

so the claim holds. \square

By using the lemma above we obtain a matrix representation of the map $g = (g_{\vec{x}}) : \bigoplus_{i \in I} L(b_i \vec{x}_i) \rightarrow E$. Recall that $I = \{i_1, i_2, i_3\}$ is given in ascending order. Then

$$\begin{aligned}
 g_{\vec{0}} = g_{\vec{x}_j} = \cdots = g_{a_j \vec{x}_j} &= \begin{array}{|c|c|c|} \hline I_{n+1} & 0 & I_{n+1} \\ \hline 0 & I_{n+1} & I_{n+1} \\ \hline \end{array} & \text{for } j = 1, 2, \dots, t \\
 g_{(a_j+1)\vec{x}_j} = \cdots = g_{\vec{c}} &= \begin{array}{|c|c|c|} \hline I_n & 0 & I_n \\ \hline 0 & I_n & I_n \\ \hline \end{array} & \text{for } j \notin I \\
 g_{(a_i+b_i+1)\vec{x}_i} = \cdots = g_{\vec{c}} &= \begin{array}{|c|c|c|} \hline I_n & 0 & I_n \\ \hline 0 & I_n & I_n \\ \hline \end{array} & \text{for } i \in I \\
 g_{(a_{i_1}+1)\vec{x}_{i_1}} = \cdots = g_{(a_{i_1}+b_{i_1})\vec{x}_{i_1}} &= \begin{array}{|c|c|c|} \hline I_{n+1} & 0 & X_{n+1 \times n} \\ \hline 0 & I_n & I_n \\ \hline \end{array} & \text{for } i_1 \neq 2 \\
 g_{(a_j+1)\vec{x}_j} = \cdots = g_{(a_j+b_j)\vec{x}_j} &= \begin{array}{|c|c|c|} \hline I_n & 0 & I_{n+1} \\ \hline 0 & I_n & X_{n+1 \times n} \\ \hline \end{array} & \text{for } i_2 \neq 2 \\
 g_{(a_{i_3}+1)\vec{x}_{i_3}} = \cdots = g_{(a_{i_3}+b_{i_3})\vec{x}_{i_3}} &= \begin{array}{|c|c|c|} \hline -I_n & I_n & 0 \\ \hline X_{n+1 \times n} & 0 & I_{n+1} \\ \hline \end{array}.
 \end{aligned}$$

Note that in the case $i_1 = 2$ or $i_2 = 2$ we need to switch from $X_{n+1 \times n}$, to $Y_{n+1 \times n}$ in the above block matrices. From this we obtain

Proposition 5.8. *The module of type A has the following dimensional vector*

$$\left(\begin{array}{ccccccc} & & & \vdots & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ \dots & \boxed{2n+2} & \boxed{2n+1} & \dots & \boxed{2n+1} & \boxed{2n} & \dots \\ & & & & & & \\ \boxed{0} & \boxed{2n+2} & & \dots & & & \boxed{\vec{c}} \\ & & & & & & \\ & & & & & & \\ \dots & & & \boxed{2n+2} & \boxed{2n} & \dots & \\ & & & \vdots & & & \end{array} \right),$$

where $i \in I$ and $j \notin I$. Here the captions above the frames denote the vertices of the quiver and the numbers in the frames the dimensions of the vector spaces for them.

Finally we compute the matrices of the the module E , by completing the following square:

$$(\star) \quad \begin{array}{ccc} G_{\vec{y}} & \xrightarrow{G_{\alpha=\mathbb{1}}} & G_{\vec{x}} \\ g_{\vec{y}} \downarrow & & \downarrow g_{\vec{x}} \\ E_{\vec{y}} & \xrightarrow{E_{\alpha}} & E_{\vec{x}}, \end{array}$$

to a commutative diagram, for each arrow $\alpha : \vec{x} \rightarrow \vec{y}$. Since the maps G_{α} are monomorphism, each of these squares can be complete only in one way. If for some arrow $\alpha : \vec{x} \rightarrow \vec{y}$, the maps $L(b_i \vec{x}_i)_{\alpha}$ are identities for each $i \in I$ and $g_{\vec{x}} = g_{\vec{y}}$, then E_{α} is the identity map. Therefore we need only to determine matrices for the

following arrows:

$$\begin{aligned} \alpha_1^{(j)} & \quad \text{for } j = 3, 4, \dots, t \\ \alpha_{a_j+1}^{(j)} & \quad \text{for } j = 1, 2, \dots, t \\ \alpha_{a_i+b_i+1}^{(i)} & \quad \text{for } i \in I \end{aligned}$$

In the case of the arrows of the j -th arm, with $j \neq i_3$ (and arrow $\alpha_1^{(i_3)}$) we deal with the following commutative diagram

$$\begin{array}{ccc} & G_\alpha = \begin{array}{|c|c|c|} \hline D_{b_1 \times b_2} & 0 & 0 \\ \hline 0 & D_{c_1 \times c_2} & 0 \\ \hline 0 & 0 & D_{a_1 \times a_2} \\ \hline \end{array} & \\ G_{\vec{y}} \longrightarrow & & \longrightarrow G_{\vec{x}} \\ \downarrow g_{\vec{y}} = \begin{array}{|c|c|c|} \hline I_{b_2} & 0 & A_{b_2 \times a_2} \\ \hline 0 & I_{c_2} & A_{c_2 \times a_2} \\ \hline \end{array} & & \downarrow g_{\vec{x}} = \begin{array}{|c|c|c|} \hline I_{b_1} & 0 & B_{b_1 \times a_1} \\ \hline 0 & I_{c_1} & B_{c_1 \times a_1} \\ \hline \end{array} \\ E_{\vec{y}} \dashrightarrow & E_\alpha = \begin{array}{|c|c|} \hline E_{b_1 \times b_2} & E_{b_1 \times c_2} \\ \hline E_{c_1 \times b_2} & E_{c_1 \times c_2} \\ \hline \end{array} & \dashrightarrow E_{\vec{x}} \end{array}$$

Then from the commutativity of the diagram above we get $E_\alpha = \begin{array}{|c|c|} \hline D_{b_1 \times b_2} & 0 \\ \hline 0 & D_{c_1 \times c_2} \\ \hline \end{array}$.

In the case of arrow $\alpha_{a_{i_3}+1}^{(i_3)}$ we deal with a commutative diagram of the form

$$\begin{array}{ccc} & G_\alpha = \begin{array}{|c|c|c|} \hline D_{b_1 \times b_2} & 0 & 0 \\ \hline 0 & D_{c_1 \times c_2} & 0 \\ \hline 0 & 0 & D_{a_1 \times a_2} \\ \hline \end{array} & \\ G_{\vec{y}} \longrightarrow & & \longrightarrow G_{\vec{x}} \\ \downarrow g_{\vec{y}} = \begin{array}{|c|c|c|} \hline -A_{c_2 \times b_2} & I_{c_2} & 0 \\ \hline A_{a_2 \times b_2} & 0 & I_{a_2} \\ \hline \end{array} & & \downarrow g_{\vec{x}} = \begin{array}{|c|c|c|} \hline I_{b_1} & 0 & B_{b_1 \times a_1} \\ \hline 0 & I_{c_1} & B_{c_1 \times a_1} \\ \hline \end{array} \\ E_{\vec{y}} \dashrightarrow & E_\alpha = \begin{array}{|c|c|} \hline E_{b_1 \times c_2} & E_{b_1 \times a_2} \\ \hline E_{c_1 \times c_2} & E_{c_1 \times a_2} \\ \hline \end{array} & \dashrightarrow E_{\vec{x}} \end{array}$$

Then $E_\alpha = \begin{array}{|c|c|} \hline 0 & B_{b_1 \times a_1} \cdot D_{a_1 \times a_2} \\ \hline D_{c_1 \times c_2} & B_{c_1 \times a_1} \cdot D_{a_1 \times a_2} \\ \hline \end{array}$.

In the case of the arrow $\alpha_{a_{i_3}+b_{i_3}+1}^{(i_3)}$ we deal with a commutative diagram of the form

$$\begin{array}{ccc}
 & G_{\bar{\alpha}} = \begin{array}{|c|c|c|} \hline D_{b_1 \times b_2} & 0 & 0 \\ \hline 0 & D_{c_1 \times c_2} & 0 \\ \hline 0 & 0 & D_{a_1 \times a_2} \\ \hline \end{array} & \\
 G_{\bar{y}} \longrightarrow & & \longrightarrow G_{\bar{x}} \\
 \downarrow g_{\bar{y}} & & \downarrow g_{\bar{x}} \\
 \begin{array}{|c|c|c|} \hline I_{b_2} & 0 & A_{b_2 \times a_2} \\ \hline 0 & I_{c_2} & A_{c_2 \times a_2} \\ \hline \end{array} & & \begin{array}{|c|c|c|} \hline -B_{c_1 \times b_1} & I_{c_1} & 0 \\ \hline B_{a_1 \times b_1} & 0 & I_{a_1} \\ \hline \end{array} \\
 \downarrow & \dashrightarrow E_{\bar{\alpha}} = \begin{array}{|c|c|} \hline E_{c_1 \times b_2} & E_{c_1 \times c_2} \\ \hline E_{a_1 \times b_2} & E_{a_1 \times c_2} \\ \hline \end{array} & \downarrow \\
 E_{\bar{y}} & & E_{\bar{x}}
 \end{array}$$

$$\text{Then } E_{\alpha} = \begin{array}{|c|c|} \hline -B_{c_1 \times b_1} \cdot D_{b_1 \times b_2} & D_{c_1 \times c_2} \\ \hline B_{a_1 \times b_1} \cdot D_{b_1 \times b_2} & 0 \\ \hline \end{array}.$$

Theorem 5. *The extension module of Type A can be established by the following vector spaces and matrices.*

$$\begin{array}{ccccccc}
 & \dots & & \dots & & & \dots \\
 & \swarrow & & \swarrow & & & \swarrow \\
 & \dots \xleftarrow{\mathbb{1}} k^{2n+2} \xleftarrow{E_{\alpha_{a_i+1}}^{(i)}} k^{2n+1} \xleftarrow{\mathbb{1}} \dots \xleftarrow{\mathbb{1}} k^{2n+1} \xleftarrow{E_{\alpha_{a_i+b_i+1}}^{(i)}} k^{2n} \xleftarrow{\mathbb{1}} \dots \xleftarrow{\mathbb{1}} & & & & & \dots \\
 & \swarrow & & \swarrow & & & \swarrow \\
 & k^{2n+2} \xleftarrow{E_{\alpha_1^{(i)}}} \dots \xleftarrow{E_{\alpha_1^{(j)}}} k^{2n+2} \xleftarrow{E_{\alpha_{a_j+1}}^{(j)}} k^{2n} \xleftarrow{\dots} \dots \xleftarrow{\dots} k^{2n} & & & & & k^{2n} \\
 & \swarrow & & \swarrow & & & \swarrow \\
 & \dots & & \dots & & & \dots
 \end{array}$$

where

$$\begin{aligned}
 E_{\alpha_1^{(1)}} &= E_{\alpha_1^{(2)}} = \mathbb{1}, & E_{\alpha_1^{(j)}} &= \begin{array}{|c|c|} \hline Z_{n+1}(\lambda_j) & 0 \\ \hline 0 & Z_{n+1}(\lambda_j) \\ \hline \end{array} \quad \text{for } j > 2 \\
 E_{\alpha_{a_{i_1}+1}^{(i_1)}} &= E_{\alpha_{a_{i_2}+1}^{(i_2)}} = \begin{array}{|c|c|} \hline I_{n+1} & 0 \\ \hline 0 & X_{n+1 \times n} \\ \hline \end{array}, & E_{\alpha_{a_{i_1}+b_{i_1}+1}^{(i_1)}} &= E_{\alpha_{a_{i_2}+b_{i_2}+1}^{(i_2)}} = \begin{array}{|c|c|} \hline X_{n+1 \times n} & 0 \\ \hline 0 & I_n \\ \hline \end{array}, \\
 E_{\alpha_{a_{i_3}+1}^{(i_3)}} &= \begin{array}{|c|c|} \hline 0 & I_{n+1} \\ \hline X_{n+1 \times n} & I_{n+1} \\ \hline \end{array}, & E_{\alpha_{a_{i_3}+b_{i_1}+1}^{(i_3)}} &= \begin{array}{|c|c|} \hline -I_n & I_n \\ \hline X_{n+1 \times n} & 0 \\ \hline \end{array}, \\
 E_{\alpha_{a_j+1}^{(j)}} &= \begin{array}{|c|c|} \hline X_{n+1 \times n} & 0 \\ \hline 0 & X_{n+1 \times n} \\ \hline \end{array} & & \text{for } j \notin \{i_1, i_2, i_3\}.
 \end{aligned}$$

In the case of the second arm we need to switch from the matrices X_{**} to Y_{**} . Moreover, if $a_i = 0$, then the arrow $\alpha_1^{(i)}$ coincides with the arrow $\alpha_{a_i+1}^{(i)}$, and in this case in the place of $\alpha_1^{(i)}$ we put the composition of the above matrices $E_{\alpha_1^{(i)}}$ and $E_{\alpha_{a_i+1}^{(i)}}$.

5.3. The modules of the type B_3 . This case is similar to the previous computation. We will point out the differences using the same notations as before. In the case of type B_3 we assume that

$$a_{i_3} + b_{i_3} \geq p_{i_3} \quad \text{and} \quad a_i + b_i < p_i \quad \text{for } i = i_1, i_2.$$

Let $c_{i_3} := a_{i_3} + b_{i_3} - p_{i_3}$, then $0 \leq c_{i_3} < a_{i_3}$.

The maps $x_i^{b_i} : \mathcal{O}(\vec{y}) \longrightarrow \mathcal{O}(\vec{y} + b_i \vec{x}_i)$ for $i = i_1$ or $i = i_2$ are the same as before. The map $x_{i_3}^{b_{i_3}} : \mathcal{O}(\vec{y}) \longrightarrow \mathcal{O}(\vec{y} + b_{i_3} \vec{x}_{i_3})$ has the form $\mu_{i_3} \cdot (h_{\vec{x}})_{0 \leq \vec{x} \leq \vec{c}}$, where for $j \neq i_3$ we have

$$\begin{aligned} h_0 &= h_{\vec{x}_j} = \cdots = h_{a_j \vec{x}_j} = Z_{n+2 \times n+1}(-\lambda_{i_3}) \\ h_{(a_j+1)\vec{x}_j} &= \cdots = h_{\vec{c}} = Z_{n+1 \times n}(-\lambda_{i_3}) \end{aligned}$$

and for i_3 holds

$$\begin{aligned} h_0 &= h_{\vec{x}_{i_3}} = \cdots = h_{c_{i_3} \vec{x}_{i_3}} = Z_{n+2 \times n+1}(-\lambda_{i_3}) \\ h_{(c_{i_3}+1)\vec{x}_{i_3}} &= \cdots = h_{a_{i_3} \vec{x}_{i_3}} = Z_{n+1}(-\lambda_{i_3}) \\ h_{(a_{i_3}+1)\vec{x}_{i_3}} &= \cdots = h_{\vec{c}} = Z_{n+1 \times n}(-\lambda_{i_3}) \end{aligned}$$

Then the map $g = (g_{\vec{x}})_{0 \leq \vec{x} \leq \vec{c}} : G \longrightarrow E$ has the following shape:

$$\begin{aligned} g_{\vec{0}} = g_{\vec{x}_j} = \cdots = g_{a_j \vec{x}_j} &= \begin{array}{|c|c|c|} \hline -I_{n+1} & I_{n+1} & 0 \\ \hline Z_{n+2 \times n+1}(-\lambda_{i_3}) & 0 & I_{n+2} \\ \hline \end{array} & \text{for } j \neq i_3 \\ g_{(a_j+1)\vec{x}_j} = \cdots = g_{\vec{c}} &= \begin{array}{|c|c|c|} \hline -I_n & I_n & 0 \\ \hline Z_{n+1 \times n}(-\lambda_{i_3}) & 0 & I_{n+1} \\ \hline \end{array} & \text{for } j \notin I \\ g_{(a_{i_1}+1)\vec{x}_{i_1}} = \cdots = g_{(a_{i_1}+b_{i_1})\vec{x}_{i_1}} &= \begin{array}{|c|c|c|} \hline I_{n+1} & -X_{n+1 \times n} & 0 \\ \hline 0 & Z_{n+1 \times n}(-\lambda_{i_3}) & I_{n+1} \\ \hline \end{array} & \text{for } i_1 \neq 2 \\ g_{(a_i+b_i+1)\vec{x}_i} = \cdots = g_{\vec{c}} &= \begin{array}{|c|c|c|} \hline -I_n & I_n & 0 \\ \hline Z_{n+1 \times n}(-\lambda_{i_3}) & 0 & I_{n+1} \\ \hline \end{array} & \text{for } i \in \{i_1, i_2\} \\ g_{(a_{i_2}+1)\vec{x}_{i_2}} = \cdots = g_{(a_{i_2}+b_{i_2})\vec{x}_{i_2}} &= \begin{array}{|c|c|c|} \hline -X_{n+1 \times n} & I_{n+1} & 0 \\ \hline Z_{n+1 \times n}(-\lambda_{i_3}) & 0 & I_{n+1} \\ \hline \end{array} & \text{for } i_2 \neq 2 \\ g_{\vec{0}} = g_{\vec{x}_{i_3}} = \cdots = g_{c_{i_3} \vec{x}_{i_3}} &= \begin{array}{|c|c|c|} \hline -I_{n+1} & I_{n+1} & 0 \\ \hline Z_{n+2 \times n+1}(-\lambda_{i_3}) & 0 & I_{n+2} \\ \hline \end{array} \\ g_{(c_{i_3}+1)\vec{x}_{i_3}} = \cdots = g_{a_{i_3} \vec{x}_{i_3}} &= \begin{array}{|c|c|c|} \hline -I_{n+1} & I_{n+1} & 0 \\ \hline Z_{n+1}(-\lambda_{i_3}) & 0 & I_{n+1} \\ \hline \end{array} \\ g_{(a_{i_3}+1)\vec{x}_{i_3}} = \cdots = g_{\vec{c}} &= \begin{array}{|c|c|c|} \hline -I_n & I_n & 0 \\ \hline Z_{n+1 \times n}(-\lambda_{i_3}) & 0 & I_{n+1} \\ \hline \end{array} \end{aligned}$$

Lets remark, that if $i_k = 2$ for same $k = 1, 2$ or 3 , then we need to switch matrices $X_{*,*}$ to $Y_{*,*}$ in i_k -arm.

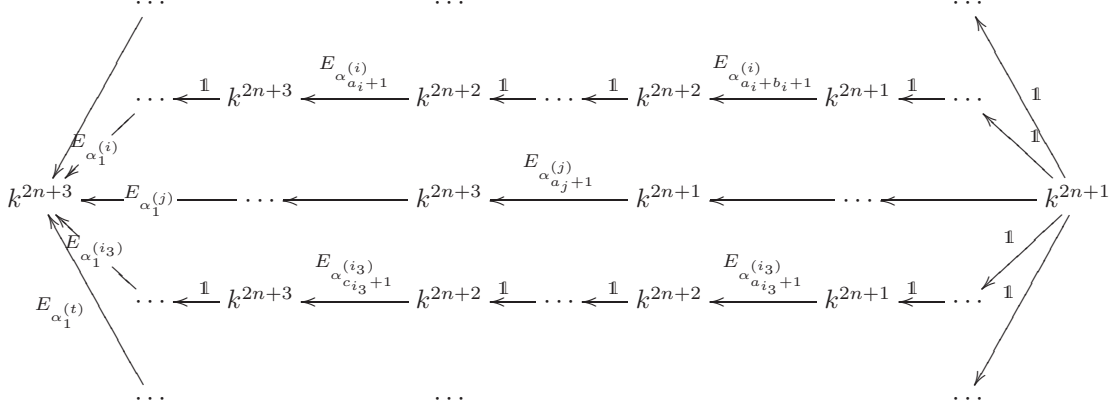
Proposition 5.9. *The module of type B_3 has the following dimension vector.*

$$\left(\begin{array}{cccccccc} & & & \cdots & & & & \\ & \vec{x}_i & a_i \vec{x}_i & (a_i+1)\vec{x}_i & (a_i+b_i)\vec{x}_i & (a_i+b_i+1)\vec{x}_i & (p_i-1)\vec{x}_i & \\ & \boxed{2n+3} & \cdots \boxed{2n+3} & \boxed{2n+3} & \cdots \boxed{2n+2} & \boxed{2n+2} & \cdots \boxed{2n+1} & \\ & & & \cdots & & & & \\ 0 & & & a_j \vec{x}_j & (a_j+1)\vec{x}_j & & & \vec{c} \\ \boxed{2n+3} & \cdots & & \boxed{2n+3} & \boxed{2n+1} & \cdots & & \boxed{2n+1} \\ & & & \cdots & & & & \\ & \vec{x}_{i_3} & c_{i_3} \vec{x}_{i_3} & (c_{i_3}+1)\vec{x}_{i_3} & (a_{i_3})\vec{x}_{i_3} & (a_{i_3}+1)\vec{x}_{i_3} & (p_{i_3}-1)\vec{x}_{i_3} & \\ & \boxed{2n+3} & \cdots \boxed{2n+3} & \boxed{2n+3} & \cdots \boxed{2n+2} & \boxed{2n+2} & \cdots \boxed{2n+1} & \\ & & & \cdots & & & & \end{array} \right),$$

where $i \in \{i_1, i_2\}$ and $j \notin I$. Here the captions above the frames denote the vertices of the quiver and the numbers in the frames the dimensions of the vector spaces for them.

Proceeding as in the case before we get representations for a module of type \mathbf{B}_3 .

Theorem 6. *The extension module E of type \mathbf{B}_3 can be exhibited by the following vector spaces and matrices.*



for $j \notin \{i_1, i_2, i_3\}$, where

$$\begin{aligned}
 E_{\alpha_1^{(j)}} &= \begin{array}{|c|c|} \hline Z_{n+1}(\lambda_j) & 0 \\ \hline 0 & Z_{n+2}(\lambda_j) \\ \hline \end{array} \text{ for } j > 2 & E_{\alpha_1^{(1)}} = E_{\alpha_1^{(2)}} = \mathbb{1}, \\
 E_{\alpha_{a_{i_1}+1}^{(i_1)}} &= \begin{array}{|c|c|} \hline -I_{n+1} & 0 \\ \hline Z_{n+2 \times n+1}(-\lambda_{i_3}) & X_{n+2 \times n+1} \\ \hline \end{array}, & E_{\alpha_{a_{i_1}+b_{i_1}+1}^{(i_1)}} &= \begin{array}{|c|c|} \hline -X_{n+1 \times n} & 0 \\ \hline Z_{n+1 \times n}(-\lambda_{i_3}) & I_{n+1} \\ \hline \end{array}, \\
 E_{\alpha_{a_{i_2}+1}^{(i_2)}} &= \begin{array}{|c|c|} \hline I_{n+1} & 0 \\ \hline 0 & X_{n+2 \times n+1} \\ \hline \end{array}, & E_{\alpha_{a_{i_2}+b_{i_2}+1}^{(i_2)}} &= \begin{array}{|c|c|} \hline X_{n+1 \times n} & 0 \\ \hline 0 & I_{n+1} \\ \hline \end{array}, \\
 E_{\alpha_{c_{i_3}+1}^{(i_3)}} &= \begin{array}{|c|c|} \hline I_{n+1} & 0 \\ \hline 0 & X_{n+2 \times n+1} \\ \hline \end{array}, & E_{\alpha_{a_{i_3}+1}^{(i_3)}} &= \begin{array}{|c|c|} \hline X_{n+1 \times n} & 0 \\ \hline 0 & I_{n+1} \\ \hline \end{array}, \\
 E_{\alpha_{a_j+1}^{(j)}} &= \begin{array}{|c|c|} \hline X_{n+1 \times n} & 0 \\ \hline 0 & X_{n+2 \times n+1} \\ \hline \end{array} & & \text{for } j \notin \{i_1, i_2, i_3\}.
 \end{aligned}$$

In the case of the second arm we need to switch from the matrices X_{**} to Y_{**} . Moreover, if $a_i = 0$, then the arrow $\alpha_1^{(i)}$ coincide with the arrow $\alpha_{a_i+1}^{(i)}$, in this case in the place of $\alpha_1^{(i)}$ we put composition of matrices $E_{\alpha_1^{(i)}}$ and $E_{\alpha_{a_i+1}^{(i)}}$.

6. EXCEPTIONAL MODULES OF THE HIGHER RANK

In this paper we have focused on the case of Λ -modules of rank two. We remark that the presented construction of them by cokernels can be applied also for exceptional modules of higher rank. For this we have to consider exact sequences of the form

$$0 \longrightarrow L \xrightarrow{f} \bigoplus_{j \in J} L(b_i \vec{x}_i) \longrightarrow E \longrightarrow 0,$$

for $J \subset \{1, 2, \dots, t\}$. In this case the cokernel E is exceptional of rank $|J| - 1$. Therefore here we obtain exceptional modules of rank from 2, to $t - 1$. It is an open question whether in this way we get all exceptional modules of rank r , for $3 \leq r \leq t$.

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